Application of Modal Scaling to the Pole Selection Phase of Parameter Estimation

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ABSTRACT

Modern modal parameter estimation algorithms are frequently presented as two stage solution processes where the first stage identifies the system poles and unscaled modal vectors (participation factors) of either long or short dimension, and where the second stage identifies the scaled modal vectors (residue vectors) of generally long dimension and modal scaling. This paper explores the value of having the long dimension, scaled modal information available during the pole selection process. Among the advantages of this approach is the availability of the full length residue vector for visualization and the modal scaling in order to evaluate relative contribution and physical significance. A comparison of the residue quality for this solution approach and the dominant traditional approaches is presented. The methods and results are compared using mean phase (MP), mean phase deviation (MPD), and vector scatter plots.

Nomenclature

N_i = Number of inputs.
N_o = Number of outputs.
N_f = Number of data lines (frequencies).
N_t = Number of data lines (times).
N_e = Number of effective modal frequencies.
N_∞ = Number of theoretical modal frequencies.
N_L = Size of long dimension.
N_S = Size of short dimension.
N = Number of modal frequencies.
λ_r = Complex modal frequency (rad/sec).
$\lambda_r = \sigma_r + j\,\omega_r$
σ_r = Modal damping.
ω_r = Damped natural frequency.
s = S-Domain (Laplace) frequency variable (rad/sec).
s_i = Generalized frequency variable (rad/sec).
ω_i = Frequency (rad/sec).
A_{pqr} = Residue for output DOF p, input DOF q, mode r.
Q_r = Modal scaling for mode r.
M_r = Modal mass for mode r.
M_{A_r} = Modal A mode r.
$\{\psi\}_r$ = Scaled modal vector for mode r.
ψ_{pr} = Scaled modal coefficient for output DOF p, mode r.
$\{\hat{\psi}\}_r$ = Unscaled modal vector for mode r.
$\{L\}_r$ = Modal participation vector for mode r.
$[I]$ = Identity matrix.
$[H(\omega)]$ = Frequency response function matrix ($N_o \times N_i$)).
$R_{I_{pq}}$ = Residual inertia for output DOF p, input DOF q.
$R_{F_{pq}}$ = Residual flexibility for output DOF p, input DOF q.

1. Introduction

Scaled modal vectors are typically estimated by modern modal parameter estimation algorithms after the modal frequencies are determined. Depending on the parameter estimation method chosen to estimate the modal frequencies, there is often an initial estimate of the unscaled modal vector or a subset of the unscaled modal vector. Internal to the software implementation, choices are available to estimate the scaled modal vectors and modal scaling in a number of ways. For experimental data that does not perfectly match the theoretical requirements of the modal parameter estimation algorithms, the choice of what procedure is used to estimate the final scaled modal vectors and associated modal scaling factors will effect the final answer. In some situations, typically for noisy data or data that was difficult to acquire, the final scaled modal vector may look different from the initial unscaled modal vector when presented in a modal vector animation. While this may be confusing to the user, it is perfectly understandable since the estimates are generally least squares or weighted least squares estimates that do not constrain the solution(s) in the same way. There are several acceptable choices for estimating the scaled modal vectors and modal scaling but some may be preferable in order to reduce this apparent inconsistency.

2. Background

In order to understand why different modal vectors are estimated at various stages in the modal parameter estimation process for modern algorithms, a general overview of the process, the theory and the practical, experimental implementation is required. Complete details of this development are given in several references [1-7] and is referred to as the Unified Matrix Polynomial Approach (UMPA). In the following explanation, only the frequency response function form of the equations is used to explain the modal vector scaling issues but an equivalent explanation involving impulse response functions (IRFs) parallels this explanation, equation by equation.

To begin with, the multiple input - multiple output (MIMO) frequency response function (FRF) model that is commonly used in the initial stage of modal parameter estimation is:

$$\sum_{k=0}^{m} \left[[\alpha_k] \, (s_i)^k \right] [H(\omega_i)] = \sum_{k=0}^{n} \left[[\beta_k] \, (s_i)^k \right] [I] \tag{1}$$

In the above model, the complex-valued modal frequencies can be estimated from the eigenvalue-eigenvector problem formed from the matrix coefficient polynomial equation involving the alpha ($[\alpha_k]$) coefficient matrices. This results in a matrix coefficient polynomial equation of the following form:

$$\left| \, [\alpha_m] \, s^m + [\alpha_{m-1}] \, s^{m-1} + [\alpha_{m-2}] \, s^{m-2} + \ldots\ldots\ldots + [\alpha_0] \, \right| = 0 \tag{2}$$

When the modal frequencies are estimated from the eigenvalue-eigenvector problem that is associated with solving this matrix coefficient polynomial equation, a unique estimate of the unscaled modal vector is identified at the same time. The length or dimension of this unscaled modal vector is equal to the dimension of the square alpha coefficients which must be equal to the row dimension of the FRF data matrix in order for the matrix coefficient polynomial equation to be conformal. Normally, this row dimension associated with the FRF or IRF data matrix is assumed to be connected with the number of outputs (N_o) that were measured.

Since the data matrix (FRF or IRF) is considered to be symmetric or reciprocal, the data matrix can be transposed, switching the effective meaning of the row and column index with respect to the physical inputs and outputs.

$$[\, H(\omega_i) \,]_{N_o \times N_i} = [\, H(\omega_i) \,]^T_{N_i \times N_o} \tag{3}$$

To eliminate possible confusion, in recent explanations of modal parameter estimation algorithms, the nomenclature of the number of outputs (N_o) and number of inputs (N_i) has been replaced by the length of the long dimension of the data matrix (N_L) and the length of the short dimension (N_S) regardless of which dimension refers to the physical output or input. This means that the above reciprocity relationship can be restated as:

$$[\, H(\omega_i) \,]_{N_L \times N_S} = [\, H(\omega_i) \,]^T_{N_S \times N_L} \tag{4}$$

Note that the reciprocity relationships in Equation 3 and 4 are a function of the common degrees of freedom (DOFs) in the short and long dimensions. If there are no common DOFs, there are no reciprocity relationships. Nevertheless, the importance of Equation 3 and 4 comes from the idea that the dimensions of the FRF matrix can be transposed and this affects the size of the square alpha coefficients in the matrix coefficient polynomal equation.

Finally, once the modal frequencies and unscaled modal vectors are estimated via the eigenvalue-eigenvector problem, the residues (numerators) of the partial fraction model of the FRF data matrix are used to estimate the final, scaled modal vectors and modal scaling. Note that the unscaled modal vector found in the eigenvalue-eigenvector problem is available to be used as a weighting vector in the estimation of the residues and, therefore, the final scaled modal vectors and modal scaling. Also note that this weighting vector may be of length equal to the long or short dimension, depending on the modal parameter estimation algorithm being used.

$$[\, H(\omega_i)\,]_{N_o \times N_i} = \sum_{r=1}^{N} \frac{[\, A_r\,]_{N_o \times N_i}}{j\omega_i - \lambda_r} + \frac{[\, A_r^*\,]_{N_o \times N_i}}{j\omega_i - \lambda_r^*} = \sum_{r=1}^{2N} \frac{[\, A_r\,]_{N_o \times N_i}}{j\omega_i - \lambda_r} \qquad (5)$$

This process means that most modern parameter estimation algorithms are implemented in a two stage procedure that has three steps as follows:

Stage 1, Step 1

- Load Measured Data into Over-Determined Linear Equation Form.
  - Utilize Matrix Coefficient Polynomial Based Model (Equation 1).
  - Find Scalar or Matrix Coefficients ($[\alpha_k]$ and $[\beta_k]$).
  - Implement for Various Model Orders (Consistency/Stability Diagram).

Stage 1, Step 2

- Solve Matrix Coefficient Polynomial for Modal Frequencies (Equation 2).
  - Formulate Eigenvalue-Eigenvector Problem.
  - Eigenvalues Determine the Modal Frequencies (λ_r).
  - Eigenvectors Determine the Unscaled Modal Vectors ($\{\psi_r\}$) of dimension N_S or N_L.

Stage 2, Step 3

- Load Measured Data Into Over-Determined Linear Equation Form (Equation 5).
  - Determine Modal Vectors and Modal Scaling from Residues.

The above procedure means that most modern modal parameter estimation algorithms can be summarized by the following Table:

	Domain		Matrix Polynomial Order		Coefficients	
Algorithm	Time	Freq	Low	High	Scalar	Matrix Basis
Complex Exponential Algorithm (CEA)	•			•	•	
Least Squares Complex Exponential (LSCE)	•			•	•	
Polyreference Time Domain (PTD)	•			•		$N_S \times N_S$
Ibrahim Time Domain (ITD)	•		•			$N_L \times N_L$
Multi-Reference Ibrahim Time Domain (MRITD)	•		•			$N_L \times N_L$
Eigensystem Realization Algorithm (ERA)	•		•			$N_L \times N_L$
Polyreference Frequency Domain (PFD)		•	•			$N_L \times N_L$
Simultaneous Frequency Domain (SFD)		•	•			$N_L \times N_L$
Multi-Reference Frequency Domain (MRFD)		•	•			$N_L \times N_L$
Rational Fraction Polynomial (RFP)		•		•	•	$N_S \times N_S$
Orthogonal Polynomial (OP)		•		•	•	$N_S \times N_S$
Polyreference Least Squares Complex Frequency (PLSCF)		•		•	•	$N_S \times N_S$

TABLE 1. Summary of Modal Parameter Estimation Algorithms

2.1 Theoretical Issues

The previous section overviews the overall procedure used by most modal parameter estimation algorithms. This procedure gives rise to two separate estimates of all or portions of the normalized modal vectors that are estimated. In general, only one estimate of each complete scaled modal vector and its associated modal scaling is desired. Since two independent, least squares or weighted least squares processes are involved using potentially different portions of the data matrix, there is no constraint that assures that the different estimates of the modal vectors will be the same. This is highlighted by the following theoretical and experimental discussions.

The theoretical frequency response function (FRF) is directly related to the typical mass, damping and stiffness matrix relationships used to describe mechanical systems. This relationship for a large dimension, descretized system is normally represented by Equation 6.

$$[\,H(s_i)\,]_{N_\infty \times N_\infty} = \left[\,[\,M\,]_{N_\infty \times N_\infty} s_i^{\,2} + [\,C\,]_{N_\infty \times N_\infty} s_i^{\,1} + [\,K\,]_{N_\infty \times N_\infty} s_i^{\,0}\,\right]^{-1} \tag{6}$$

This inverse relationship indicates that the characteristics of the FRF matrix will take on the same properties as the combined mass, damping and stiffness matrices. This means that both matrix representations should yield the same (complex-valued) modal frequencies (λ_r). Since the mass, damping and stiffness matrices will be symmetric or reciprocal for almost all mechanical systems, this also means that the FRF matrix will have the reciprocity property. Frequently, the damping matrix is assumed to be proportional damping, or Rayleigh damping, (which includes the trivial sub-case of zero damping) which will mean that the modal vectors can be normalized to be a set of real-valued coefficients, referred to as normal modes, which physically indicates that the motion of the masses are limited to in and out-of-phase relationships to one another. If the damping matrix does not follow this mathematical form, then the modal vectors cannot be normalized to a set of real-valued coefficients. For this case, some of the modal coefficients must always be complex-valued, physically indicating the motion of the masses may take on any phase relationship with respect to one another.

Rather than relating the FRF matrix to unknown matrix coefficients like mass, stiffness and damping, a different equivalent representation of each term of the FRF matrix can be formulated. One common formulation is a partial fraction, pole-residue model that can be given by Equation 7. This partial fraction representation is very common in generalized descriptions of open and closed loop control systems as well as open loop structural dynamics systems. This partial fraction model form is the most frequently used model for estimating the scaled modal vectors once the complex-valued modal frequencies are known.

$$H_{pq}(s_i) = \sum_{r=1}^{N_\infty} \frac{A_{pqr}}{s_i - \lambda_r} + \frac{A_{pqr}^{*}}{s_i - \lambda_r^{*}} = \sum_{r=1}^{2N_\infty} \frac{A_{pqr}}{s_i - \lambda_r} \tag{7}$$

The above partial fraction model for one FRF can be generalized into the matrix model in the following way:

$$[\,H(s_i)\,]_{N_\infty \times N_\infty} = \sum_{r=1}^{N_\infty} \frac{[\,A_r\,]_{N_\infty \times N_\infty}}{s_i - \lambda_r} + \frac{[\,A_r^{*}\,]_{N_\infty \times N_\infty}}{s_i - \lambda_r^{*}} = \sum_{r=1}^{2N_\infty} \frac{[\,A_r\,]_{N_\infty \times N_\infty}}{s_i - \lambda_r} \tag{8}$$

In the above models, the complex-valued modal frequencies (λ_r) are the same as the those in the original mass, damping and stiffness matrix system. The normalized modal vectors can be found from the numerators of the partial fraction terms, referred to as the residues (A_{pqr}). Specifically the relationships between the residues, the normalized modal vectors and the modal scaling can be given by the following:

$$A_{pqr} = Q_r\, \psi_{pr}\, \psi_{qr} \tag{9}$$

Note that in the above equation the scaling term (Q_r) is normally related to one of two different forms of scaling that accompanies the complete normalized modal vector ($\{\psi\}_r$). For the special case where the damping is proportional, the relationship is:

$$Q_r = \frac{1}{j\, 2M_r\, \omega_r} \tag{10}$$

For the general case where the damping is either proportional or not, the relationship is:

$$Q_r = \frac{1}{M_{A_r}} \tag{11}$$

For the above general case, note that the normalized modal vector cannot be a real-valued vector and must contain some complex-valued coefficients.

With these relationships in mind, the partial fraction model can be manipulated into several other common forms of the equation that is used to estimate the complex-valued modal vectors once the modal frequencies are known. For example:

$$[H(s_i)]_{N_\infty \times N_\infty} = [\psi\,]_{N_\infty \times 2N_\infty} \left\lceil \frac{Q_r}{s_i - \lambda_r} \right\rfloor_{2N_\infty \times 2N_\infty} [\psi\,]^T{}_{2N_\infty \times N_\infty} \tag{12}$$

2.2 Experimental Issues

When experimental data is acquired, generally the data matrix will be rectangular limited by the number of inputs and outputs. Likewise, the data will be taken at discrete frequencies, generally equally spaced, along the frequency axis, not throughout the S – Domain as Equations 6-8 imply. Therefore, the above Equation 7 can be restated as follows:

$$H_{pq}(\omega_i) = \sum_{r=1}^{N_\infty} \frac{A_{pqr}}{j\omega_i - \lambda_r} + \frac{A^*_{pqr}}{j\omega_i - \lambda^*_r} = \sum_{r=1}^{2N_\infty} \frac{A_{pqr}}{j\omega_i - \lambda_r} \tag{13}$$

Equation 12 can then be restated as:

$$[H(\omega_i)]_{N_L \times N_S} = [\psi\,]_{N_L \times 2N} \left\lceil \frac{Q_r}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} [\psi\,]^T{}_{2N \times N_S} \tag{14}$$

Or, in transposed form:

$$[H(\omega_i)]_{N_S \times N_L} = [\psi\,]_{N_S \times 2N} \left\lceil \frac{Q_r}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} [\psi\,]^T{}_{2N \times N_L} \tag{15}$$

In order to clearly indicate how the unscaled and scaled modal vectors participate in the solution, the above two equations can be reformulated as follows:

$$[H(\omega_i)]_{N_L \times N_S} = [\hat{\psi}\,]_{N_L \times 2N} \left\lceil \frac{1}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} \lceil Q_r \rfloor_{2N \times 2N} [\psi\,]^T{}_{2N \times N_S} \tag{16}$$

$$[H(\omega_i)]_{N_S \times N_L} = [\hat{\psi}\,]_{N_S \times 2N} \left\lceil \frac{1}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} \lceil Q_r \rfloor_{2N \times 2N} [\psi\,]^T{}_{2N \times N_L} \tag{17}$$

The modal scaling (Q_r) and the modal vectors ($\{\psi\}_r$) are normally combined into a single scaled modal participation vector ($\{L\}_r$) for each mode yielding:

$$[H(\omega_i)]_{N_L \times N_S} = [\hat{\psi}\,]_{N_L \times 2N} \left\lceil \frac{1}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} [L\,]^T{}_{2N \times N_S} \tag{18}$$

$$[H(\omega_i)]_{N_S \times N_L} = [\hat{\psi}\,]_{N_S \times 2N} \left\lceil \frac{1}{j\omega_i - \lambda_r} \right\rfloor_{2N \times 2N} [L\,]^T{}_{2N \times N_L} \tag{19}$$

In Equations 18 and 19, the number of modal frequencies (N), the modal frequencies (λ_r) and the unscaled modal vectors ([$\hat{\psi}$]) are estimated in the first stage of the modal parameter estimaton procedure resulting from the eigenvalue-eigenvector problem. The modal participation vectors ([L]) are estimated in the second stage of the modal parameter estimation procedure. Finally, the residues (A_{pqr}) with appropriate units (displacement/force $\times$ rad/sec) are estimated from the following relationship:

$$A_{pqr} = \hat{\psi}_{pr} \, L_{qr} \tag{20}$$

Modal scaling (Q_r) and the scaled modal vectors ([ψ]) for the long dimension are then estimated appropriately from the residue vectors ({A_r}) reconstructed for the long dimension using the above equation. Finally, modal mass (M_r) or modal A (M_{A_r}) are estimated from scaled modal vectors ([ψ]) and Equations 9 and 10 or Equations 9 and 11, if needed.

Therefore, the normalized, scaled modal vectors can be different from the normalized, unscaled modal vectors for several reasons. First of all, if the data that is taken has noise, nonlinearities or is not reciprocal (or in other words has realistic problems), the data does not match the primary assumptions relating the experimental data to the model. This may result in different vectors since there is a mismatch between the data and the model. Second, the unscaled modal vectors will generally be of different dimension (short versus long or long versus short) when compared to each other. If the unscaled modal vector is of long dimension, all or some subset of the unscaled modal vector can be used as weighting in the second phase when the scaled modal vector is estimated. The two different stages of the modal parameter estimation procedure do not constrain the unscaled and scaled modal vectors to have the same normalized shape. Since the unscaled modal vector may be completely, partially or not involved in the weighting of the scaled modal vector, different scaled modal vectors may result. At the matching degree of freedoms, the modal coefficients should be the same if the system is reciprocal but this is not a constraint in the estimation process.

2.3 Residuals

To account for the modal contribution represented by the modes with natural frequencies below and above the frequency range of interest (F_{Min} to F_{Max}), it is common for *residual terms* to be included to account for the effect of these modes within the frequency range of interest. The basic partial fraction equation can be rewritten for a single frequency response function as:

$$H_{pq}(\omega) = R_{F_{pq}} + \sum_{r=1}^{n} \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r} + R_{I_{pq}}(\omega) \tag{21}$$

where:

- $R_{F_{pq}}$ = Residual flexibility
- $R_{I_{pq}}(s)$ = Residual inertia

The residual term that compensates for modes below the minimum frequency of interest is called the *inertia restraint*, or *residual inertia*. The residual term that compensates for modes above the maximum frequency of interest is called the *residual flexibility*. These residuals are a function of each frequency response function measurement and are not global properties of the frequency response function matrix. One example of this common form of residuals is shown graphically in Figure 1 where the frequency band of interst is from 25 Hz. to 175 Hz. The blue curve represents the residual effect of the modes below the frequency band of interest (inertia restraint) and the red curve represents the residual effect of the modes above the frequency range of interest (residual flexibility). Note that in this example, both residuals will be dominantly real-valued.

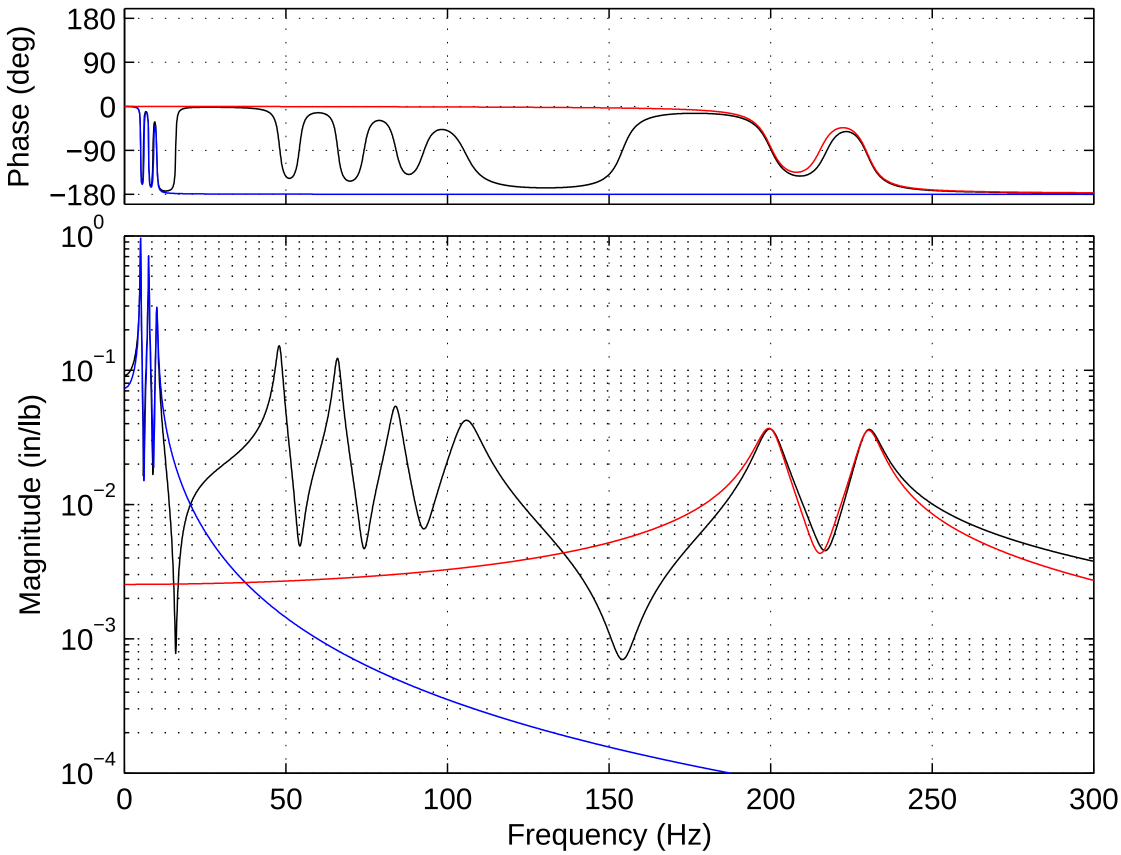

Figure 1. Graphical Example of Residual Contributions

Note that residuals must be added for every measurement that participates in the weighted estimation of the scaled modal vectors. For cases where the unscaled modal vector is estimated for the long dimension in the first stage of the modal parameter estimation, this means that a large number of unknowns representing the residuals may need to be added when only a limited number of modal participation coefficients are needed for the short dimension. This can cause memory issues and may degrade the estimates of the scaled modal vectors.

It should also be recognized that another minor source of unconstrained variability with respect to the scaled modal vectors comes from the choice of the residual model (one, two or more residual terms) and the frequency range of interest. Since residuals are effectively mathematical terms added to the equation set in an attempt to accommodate potential effects of out-of-band modes, except under specific circumstances, these terms do not have physical significance. It is also important to note that residuals are somtimes computed after the residues are estimated and have no affect on the residues at all. Further, because any of the various residual models can be applied to each of the following scaled modal vector formulations, this paper is not going to explore the independent influence of residuals upon the resultant scaled modal vector quality. The reader interested in the various residual models is encouraged to read some of the other papers on this topic [1,3-8] . Note, however, that since the objective of this paper is to present a qualitative (not quantitative) comparison of the influence of solution technique upon the computed residues, residuals will be included in some solutions in an attempt to provide the best reasonable solution for each technique.

3. Possible Scaled Vector Formulations

Based upon the previous discussions, there are at least five common implementations available for estimating the scaled modal vectors from a set of measured FRF data:

- Method 1: Single or multiple reference FRF measurements with least squares estimation of residues, measurement by measurement, with or without residuals.
- Method 2: Multiple reference FRF measurements with short dimension basis, weighted least squares solution for residues, with or without residuals.
- Method 3: Multiple reference FRF measurements with short dimension basis (extracted from long dimension), weighted least squares solution for residues, with or without residuals.
- Method 4: Multiple reference FRF measurements with long dimension basis, weighted least squares solution for residues, with or without residuals.
- Method 5: Multiple reference FRF measurements with short or long dimension basis, weighted least squares solution for residues, with computational modes utilized as residuals.

Each of the above methods will give a slightly different result for a relatively consistent and noise free set of measured FRF data. If the FRF data has realistic problems caused by random or bias errors (noise), linearity, time varying and/or reciprocity issues, the results may be significantly different.

Another way of describing the various methods is in terms of how each method constrains the solution for the scaled modal vectors. In Method 1, the only constraint is that each FRF measurement is estimated with the same set of modal frequencies. In Method 2, the constraints include a common set of modal frequencies and the unscaled modal vectors of length equal to the short dimension. In Method 3, the constraints include a common set of modal frequencies and a set of unscaled modal vectors of length equal to the short dimension. However, in Method 3, these unscaled modal vectors are obtained by taking the short dimension subset of the DOF(s) from the computed long dimension, unscaled modal vectors. This method is often used to maintain similarity with Method 2 and to reduce the possibility of degradation of the solution when residuals are included. In Method 4, the constraints include a common set of modal frequencies and the unscaled modal vectors of length equal to the long dimension. This method works well when residuals are not needed but has some numerical issues when residuals are included. In Method 5, the unscaled modal vectors and scaled modal vectors are estimated for all possible eigenvalues of the long or short dimension problem before the set of modal frequencies are chosen. The characterisitics of the unscaled modal vectors and the scaled modal vectors along with the consistency/stability of all of the modal parameters are used to select the set of modal parameters that is optimum. While Method 5 is not a totally new approach, with the increase in available memory and compute speed, this method is now more viable.

In order to compare the results for unscaled and scaled modal vectors estimated by different methods, a visual plot of the complex valued vectors in the complex plane is very useful. Figure 2 is an example of a comparison of the unscaled and scaled modal vectors. In order to quantify the characteristics of the vectors, the computation of mean phase and mean phase deviation or mean phase correlation is often utilized [9-13].

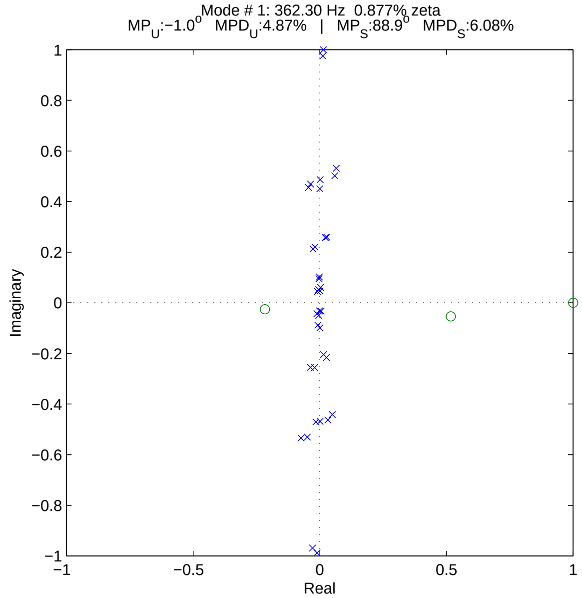

Figure 2. Typical Complex Plot of Unscaled and Scaled Modal Vectors

Mean Phase (MP or θ_{MP}): Mean phase is the average angle of the (modal) vector with respect to the real/imaginary axes. A mean phase of 0^0 means that the vector is oriented around the real axis; a mean phase of $\pm 90^0$ means that the vector is oriented around the imaginary axis. The mean phase is computed by first solving the following eigenvalue problem:

$$[\mathrm{Im}(\{\psi\})\ \mathrm{Re}(\{\psi\})]^T\ [\mathrm{Im}(\{\psi\})\ \mathrm{Re}(\{\psi\})]\ \{v\} = \bar{\lambda}\ \{v\} \tag{22}$$

Then selecting the eigenvector associated with the smallest eigenvalue and recognizing its form:

$$\{v\}_{\bar{\lambda}_{\min}} = \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \tag{23}$$

$$MP = \theta_{MP} = \tan^{-1}\left(\frac{v_2}{-v_1}\right) \tag{24}$$

Note that the above relation returns the principal angle for the vector, bounded by $\pm 90^0$. If it is desired that the angle indicate the direction of largest response, it is necessary to pay attention to the sign of the numerator and denominator of the inverse tangent function. In general, the mean phase is used to determine whether the modal vector is dominantly real-valued or imaginary-valued so this is not always done.

Mean Phase Deviation (MPD): Mean phase deviation represents the scatter of the (modal) vector about the mean phase angle on a fraction or percentage basis. A mean phase deviation of 0.0 percent means that the vector is a normal mode oriented about (rotated to) the mean phase angle. A mean phase deviation larger than 0.0 percent indicates that the vector is a complex mode oriented about the mean phase angle. The mean phase deviation is computed by the following equation:

$$MPD = \frac{\left\| \mathrm{Im}\left\{ e^{-j\,\theta_{MP}} \{\psi_r\} \right\} \right\|}{\left\| \mathrm{Re}\left\{ e^{-j\,\theta_{MP}} \{\psi_r\} \right\} \right\|} \times 100\ \% \tag{25}$$

Mean Phase Correlation (MPC): Mean phase correlation is the representation of the mean phase deviation as a zero-to-one correlation coefficient, where a value of 1.0 means that the modal vector is a normal mode. The mean phase correlation is computed by the following equation:

$$\mathrm{MPC} = 1.0 - \frac{\mathrm{MPD}}{100} \tag{26}$$

In Figure 2, the green o's represent the plot of the coefficients of the unscaled modal vector while the blue x's represent the plot of the scaled modal vectors. The mean phase and mean phase deviation of the unscaled and scaled modal vectors are given in the title of each mode vector plot. The vectors are normalized in every case so that the largest amplitude in each vector has a unity magnitude so that they can be plotted on the same scale.

4. Examples

The figures present a set of examples comparing the qualitative impact of residue solution technique on the computed residue solutions. All figures have had the maximum value of the unscaled and scaled modal vector solutions normalized to one in order to present the results on a single graphical vector scatter diagram. In evaluating the results, the concentration is upon the qualitative change in the results; the specific absolute numerical values presented are not of primary importance.

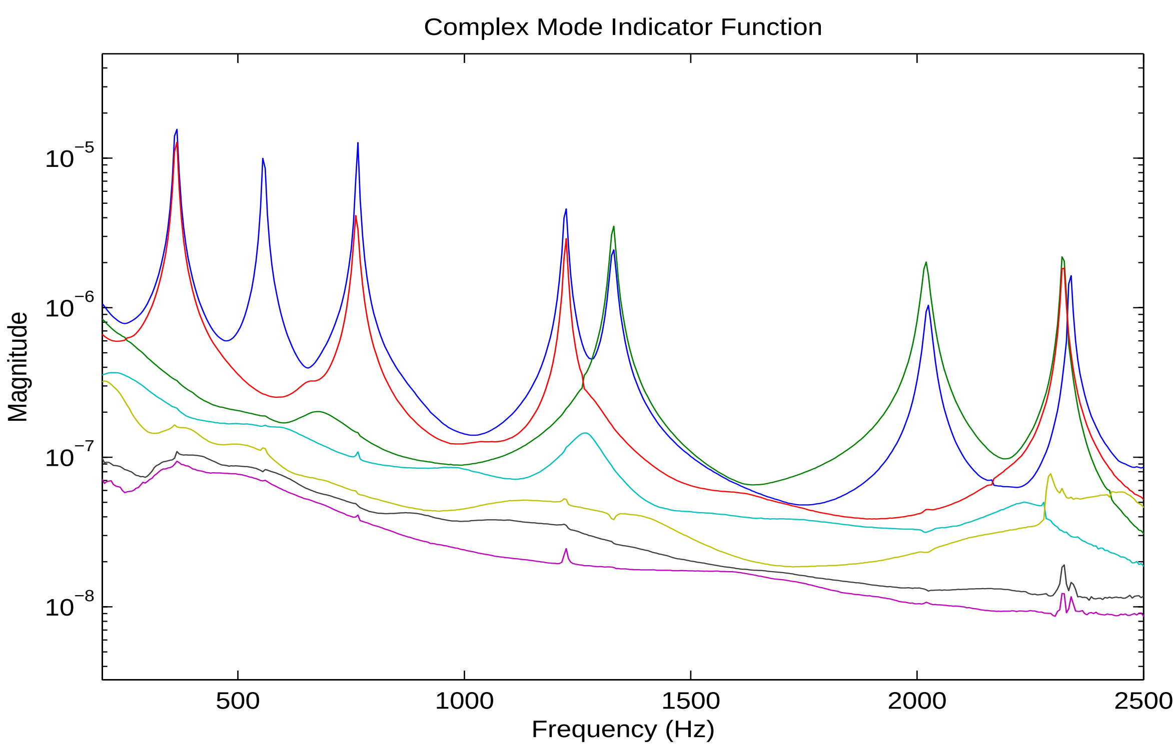

Figure 3. CMIF, C-Plate Data

Figure 3 is a Complex Mode Indicator Function (CMIF) plot of the FRF data for a 36 input (long dimension), 7 output (short dimension) matrix of FRF data [14-15]. This laboratory test structure will be used for most of the test case comparisons. The data from this test structure is particularly "clean" and yet actually exhibits some real world test issues like leakage and repeated roots (close modes).

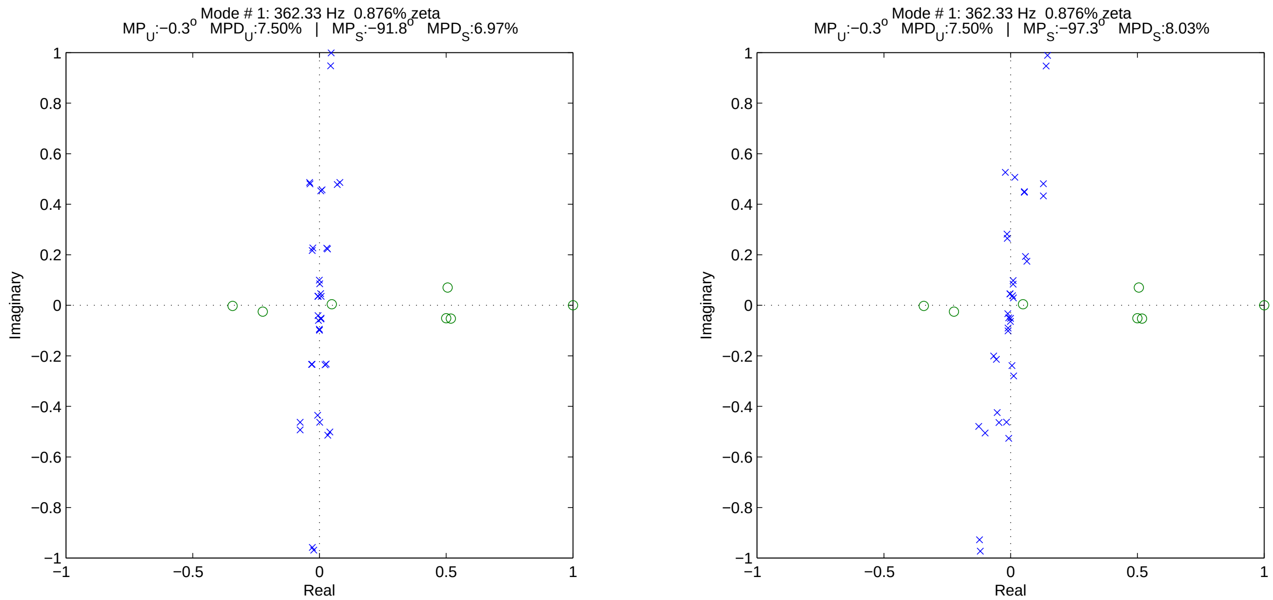

Figure 4. Vector Plots, PTD (Method 1, Different References), Phase 1 versus Phase 2, Mode 1

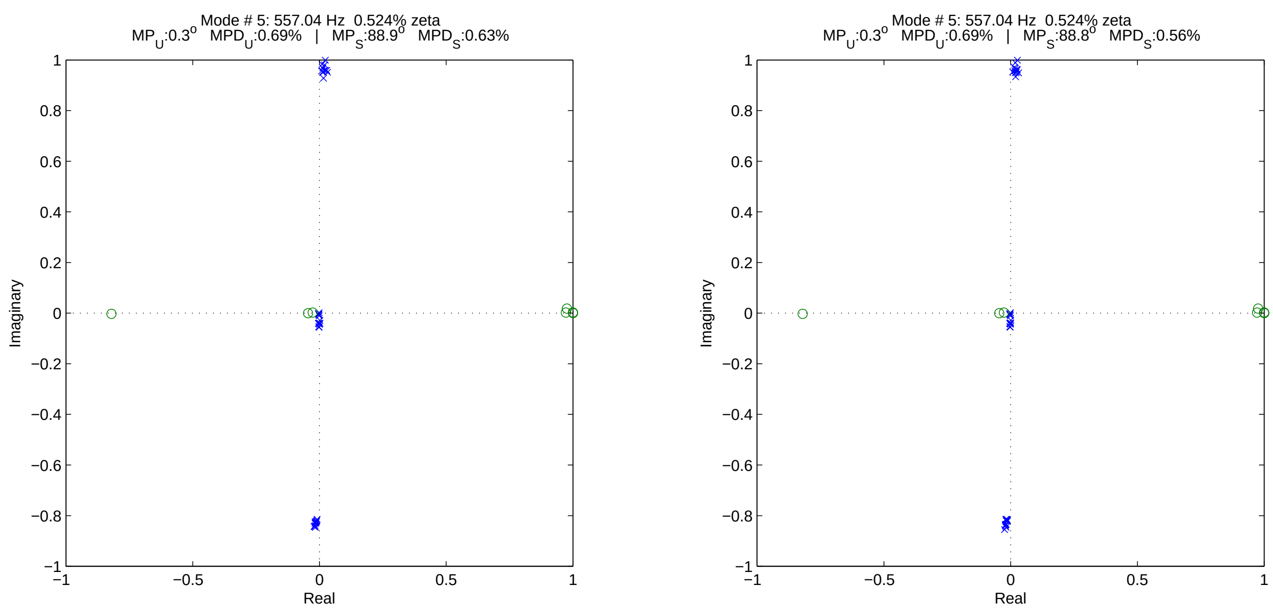

Figure 5. Vector Plots, PTD (Method 1, Different References), Phase 1 versus Phase 2, Mode 2

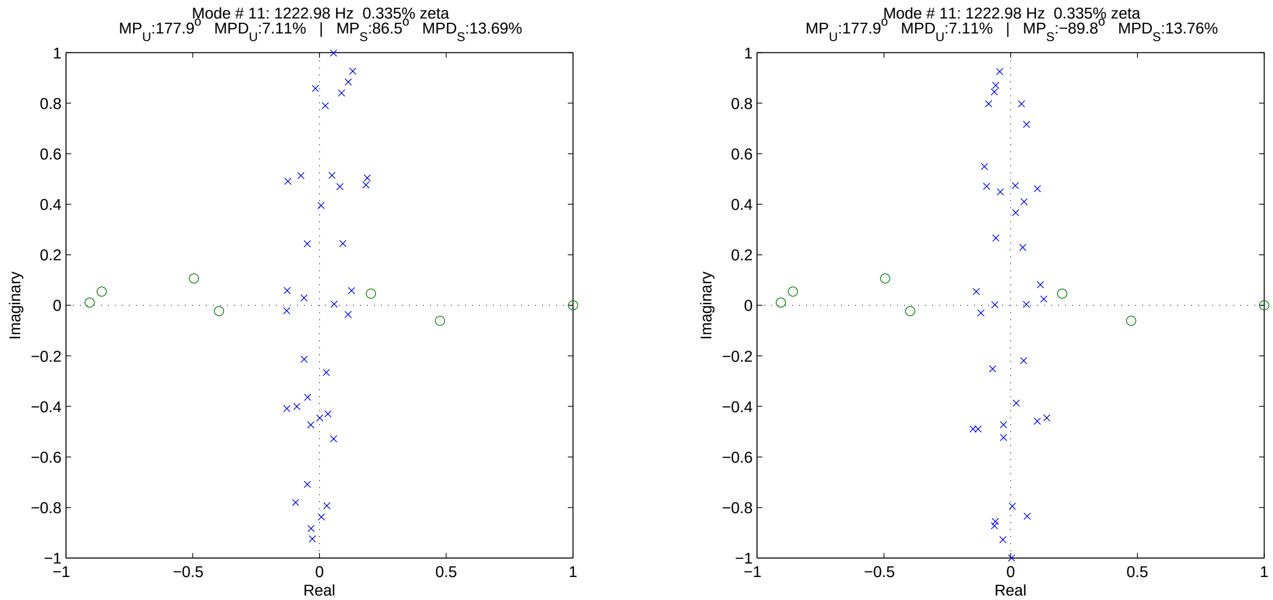

Figure 6. Vector Plots, PTD (Method 1, Different References), Phase 1 versus Phase 2, Mode 3

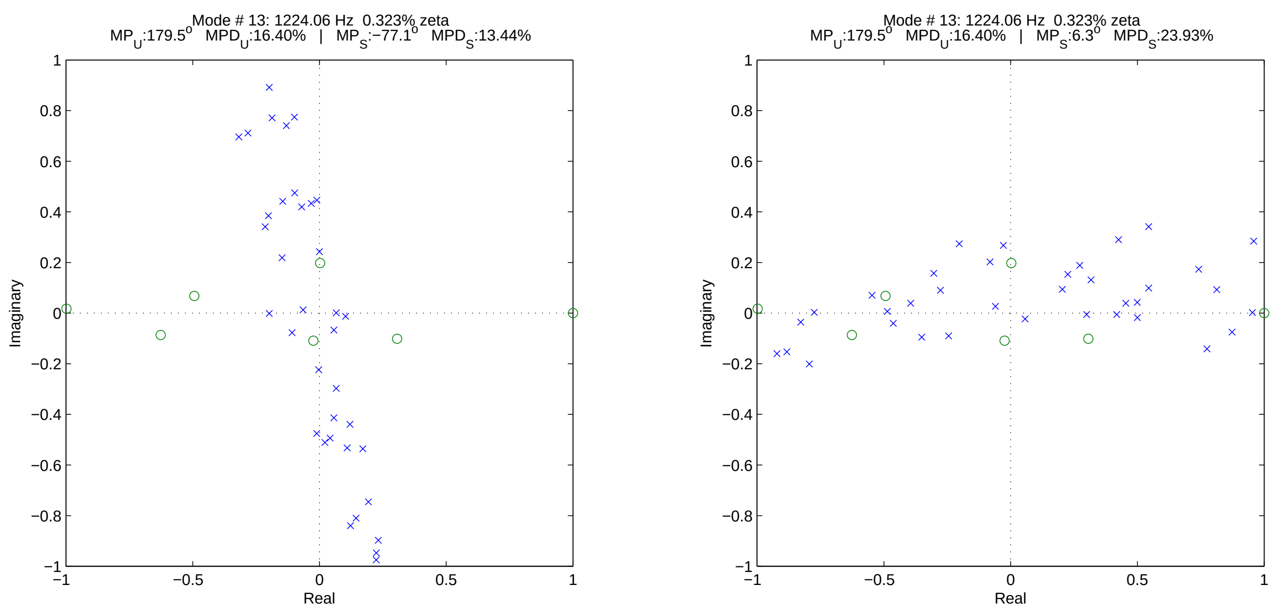

Figure 7. Vector Plots, PTD (Method 1, Different References), Phase 1 versus Phase 2, Mode 4

For Figures 4 to 7, all seven references were used in the first phase, Polyreference Time Domain (PTD), to identify the modal frequencies. Although this computed an unscaled (short dimension) modal vector, the weighting associated was not used to caluculate the residues. Instead, consistent with Method 1, using only the modal frequency information, the residues were computed from two different references independently. Comparing the results for the two different references shows that sometimes the results are quite similar, other times the results are significantly different in terms of mean phase and modal complexity, and occasionally, the results are totally distorted and inaccurate.

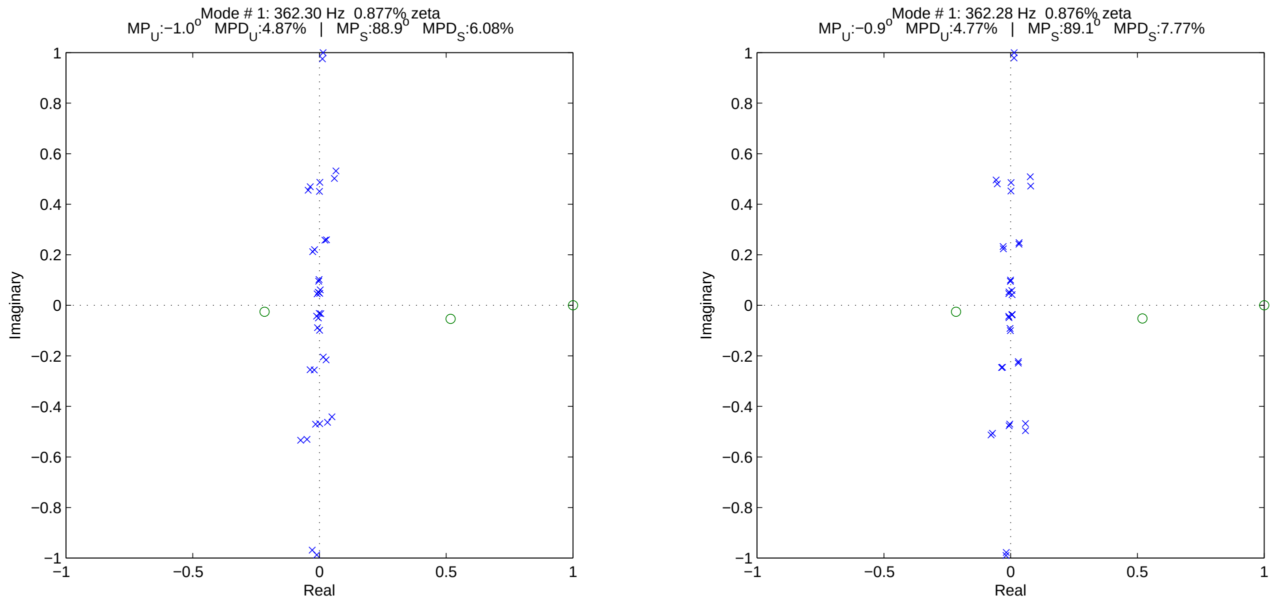

Figure 8. Vector Plots, PTD (Method 5 - left, Method 2 - right), Phase 1 versus Phase 2, Mode 1

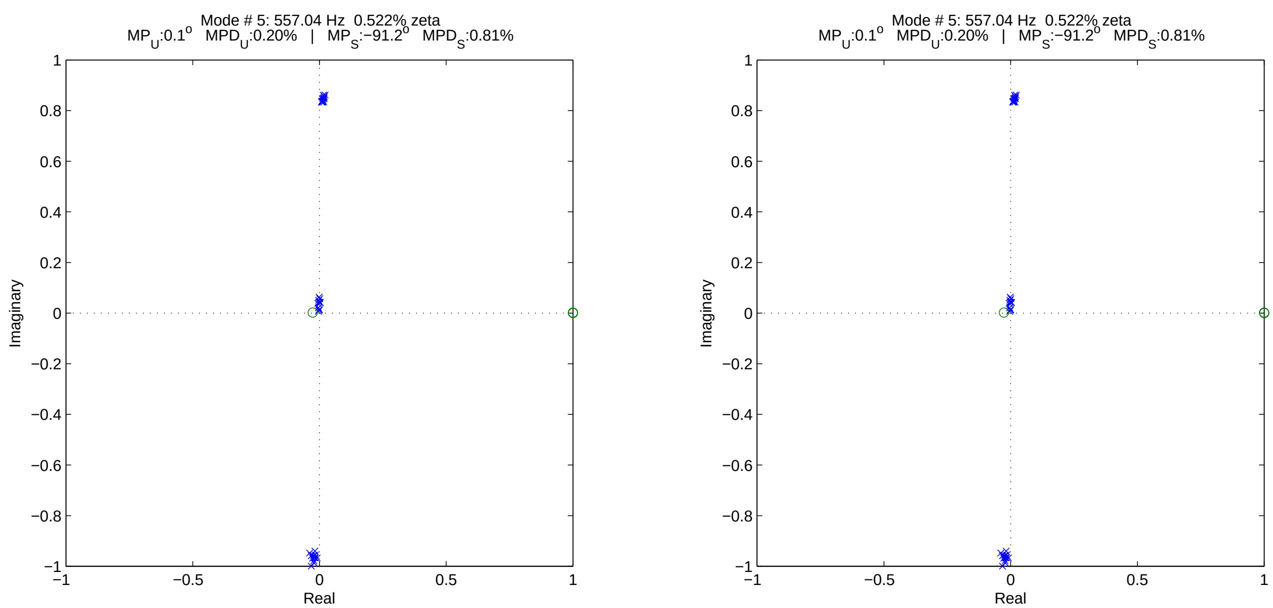

Figure 9. Vector Plots, PTD (Method 5 - left, Method 2 - right), Phase 1 versus Phase 2, Mode 2

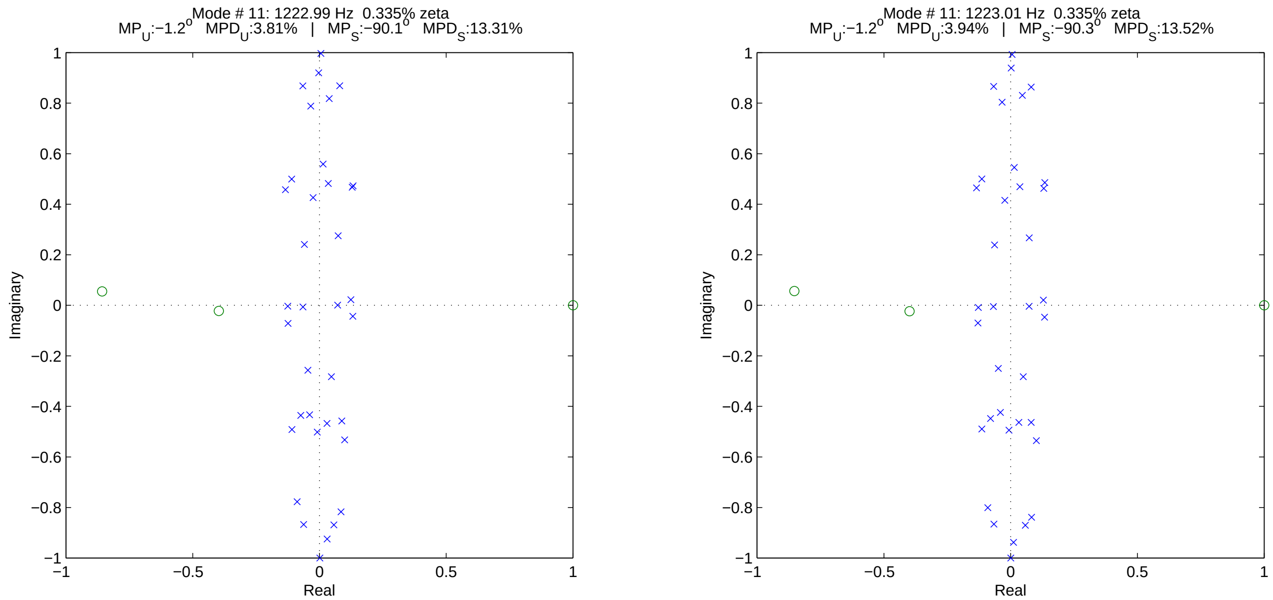

Figure 10. Vector Plots, PTD (Method 5 - left, Method 2 - right), Phase 1 versus Phase 2, Mode 3

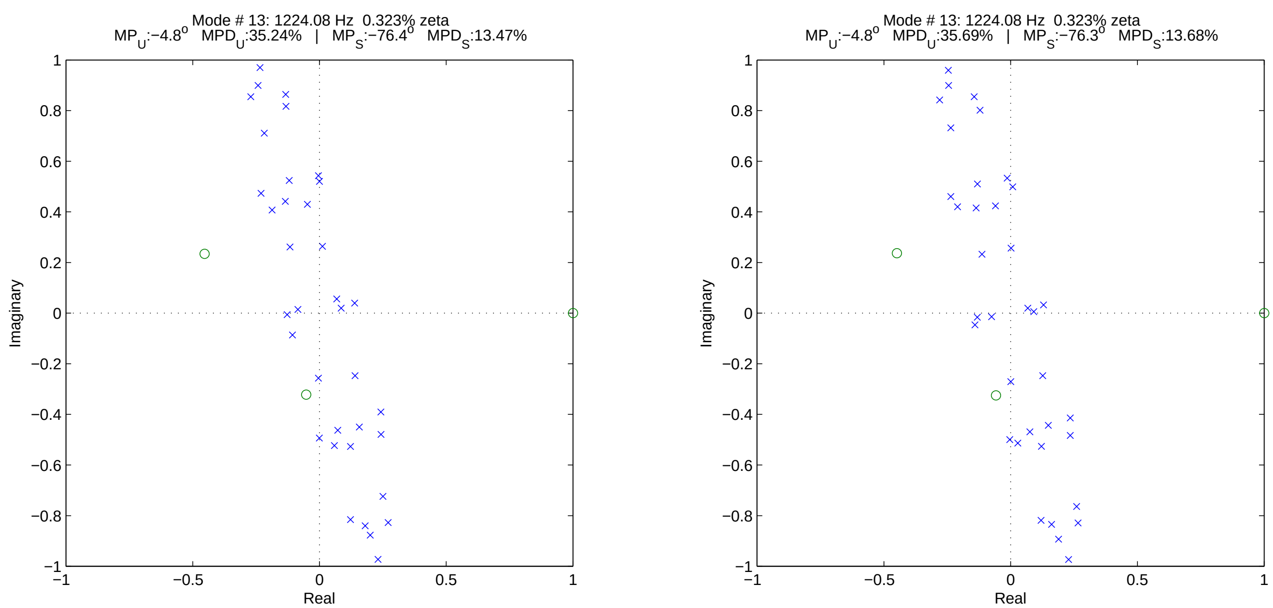

Figure 11. Vector Plots, PTD (Method 5 - left, Method 2 - right), Phase 1 versus Phase 2, Mode 4

For Figures 8 to 11, three of the potential references were used. Comparing the results of the Polyreference Time Domain (PTD) technique for Method 5 and Method 2, shows that the two techniques produce very similar results as expected since in both cases short dimension, unscaled modal vectors are computed and used to estimate the scaled (long dimension) vectors, however, having the long dimension information available for pole selection is advantageous (Method 5).

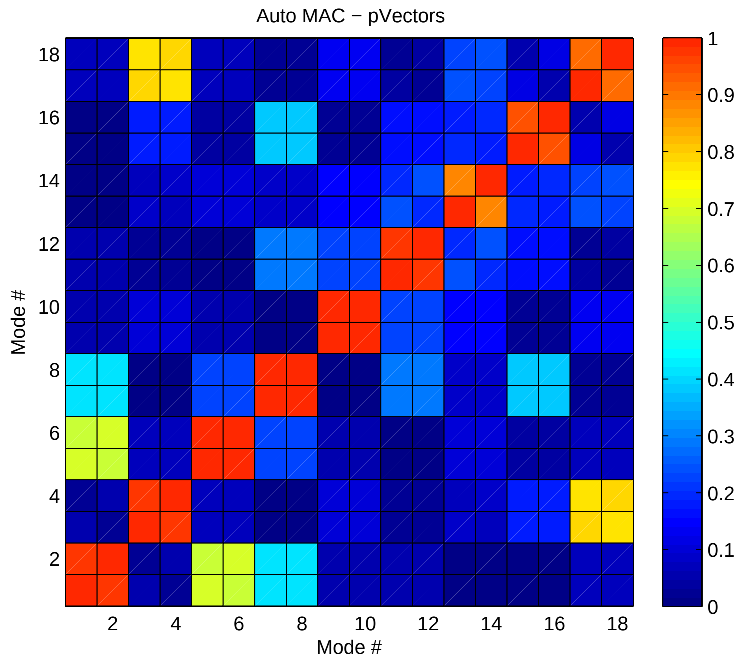

Figure 12. Auto-MAC, PTD, Unscaled Modal Vectors

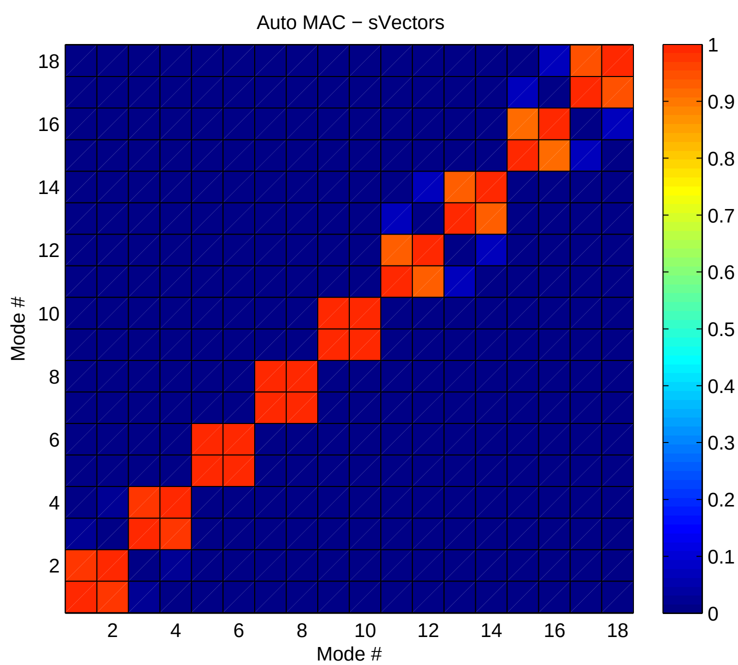

Figure 13. Auto-MAC, PTD, Scaled Modal Vectors

For the Polyreference Time Domain (PTD) technique which produces modal frequencies and a short dimension modal vector, comparing Figures 12 and 13 of the MAC for the unscaled (short dimension) and scaled (long dimension) vectors, demonstrates through the clarity of the MAC plot, the advantage of having the long dimension information available.

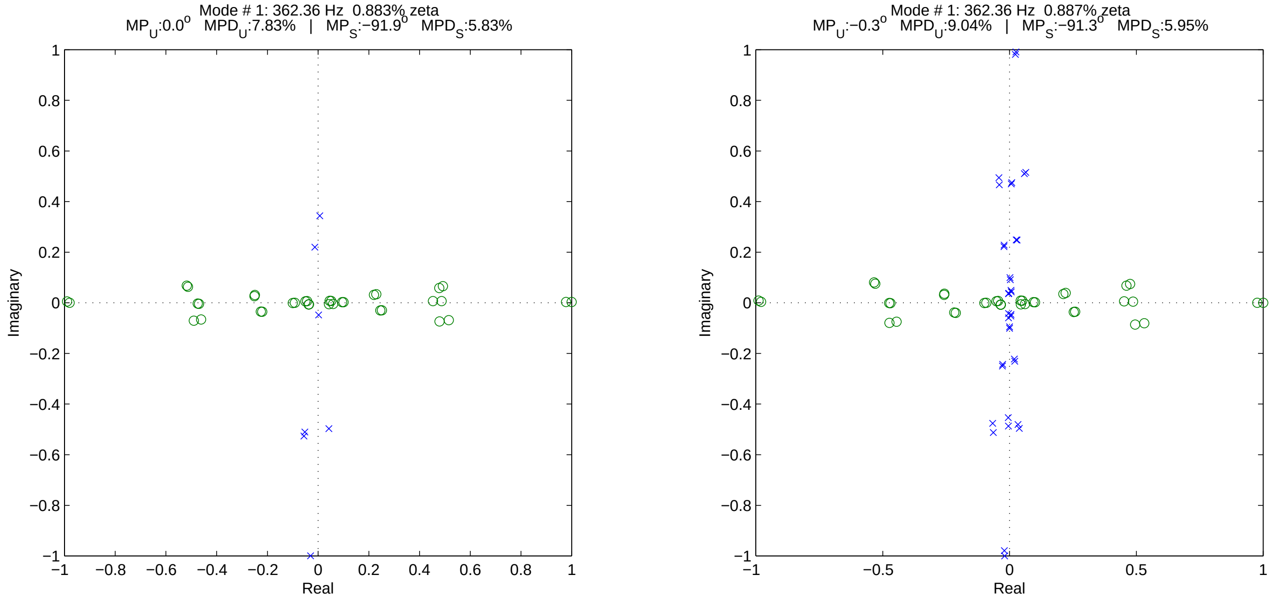

Figure 14. Vector Plots, ERA (Method 5 - left, Method 4 - right), Phase 1 versus Phase 2, Mode 1

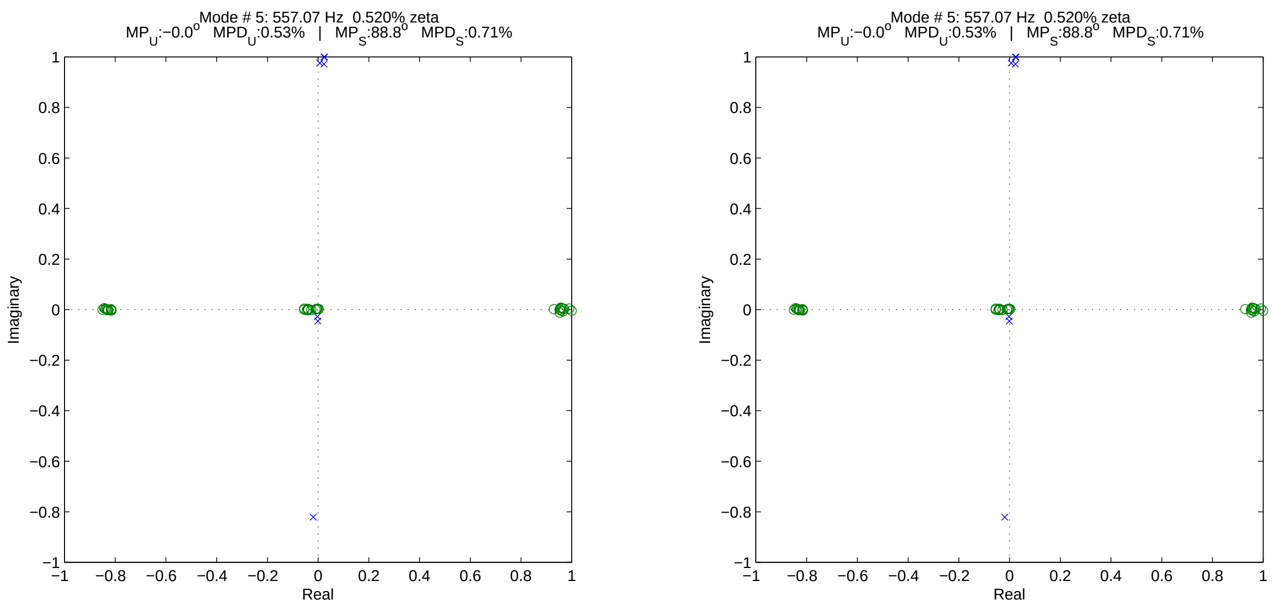

Figure 15. Vector Plots, ERA (Method 5 - left, Method 3 - right), Phase 1 versus Phase 2, Mode 2

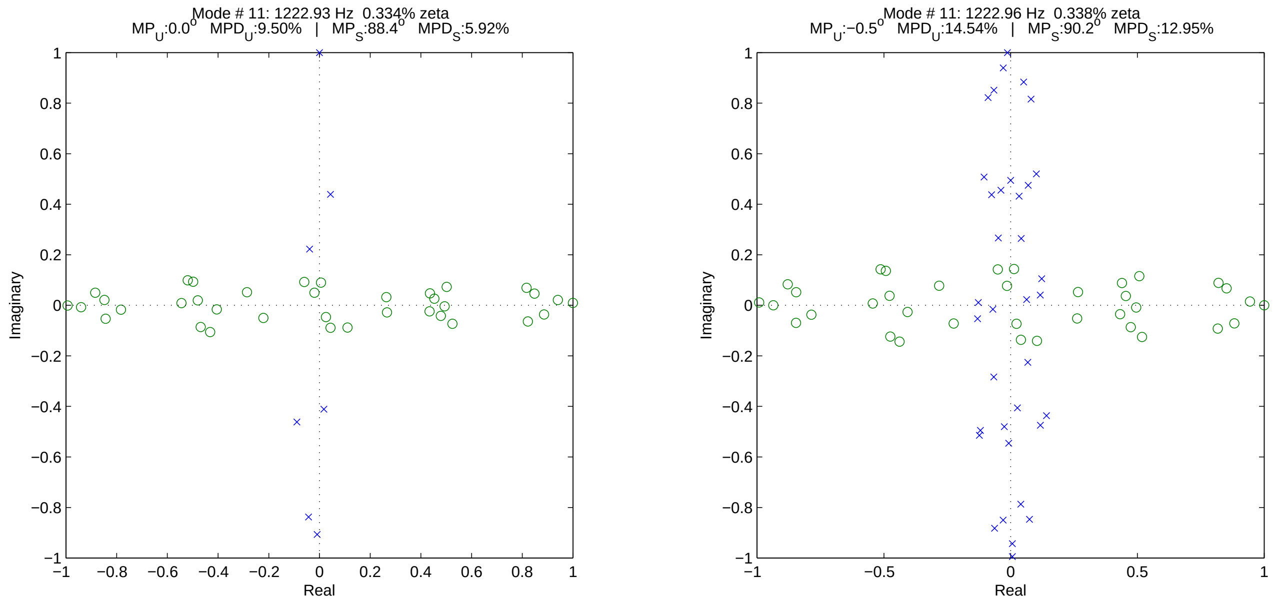

Figure 16. Vector Plots, ERA (Method 5 - left, Method 3 - right), Phase 1 versus Phase 2, Mode 3

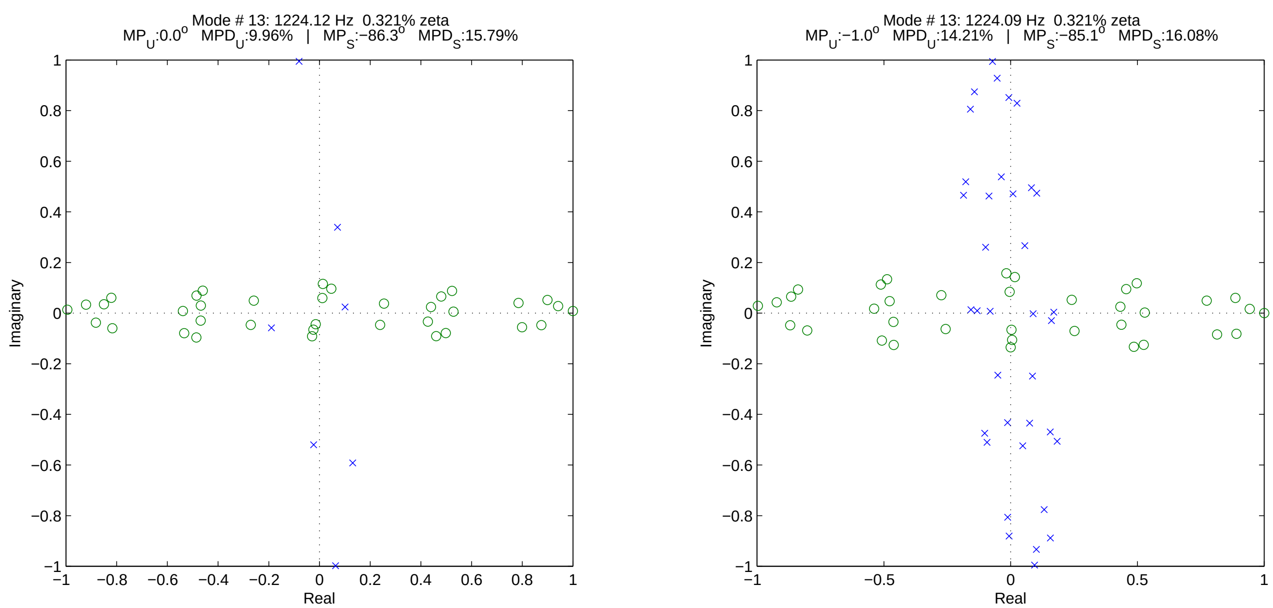

Figure 17. Vector Plots, ERA (Method 5 - left, Method 3 - right), Phase 1 versus Phase 2, Mode 4

Figures 14 to 17 show the results of using the Eigensystem Realization Algorithm (ERA) which produces long dimension, unscaled vectors. Comparing the Mean Phase Deviation (MPD) for the unscaled and the scaled modal vectors (right half of plots), shows that the computed (Method 3) scaled shape is slightly different than the first phase unscaled shape. Sometimes the computed shape is more normal, other times the computed shape is more complex. This demonstrates the unconstrained nature of the solution.

Making the same comparison on the left half of the plot (Method 5 results), shows the same expected trend. However, since the scaled vector is now along the short dimension, the content of the short dimension vectors are the modal scaling parameters for the long dimension vector. Using this solution allows retention of the viewed/computed first phase solution (long dimension, unscaled modal vector).

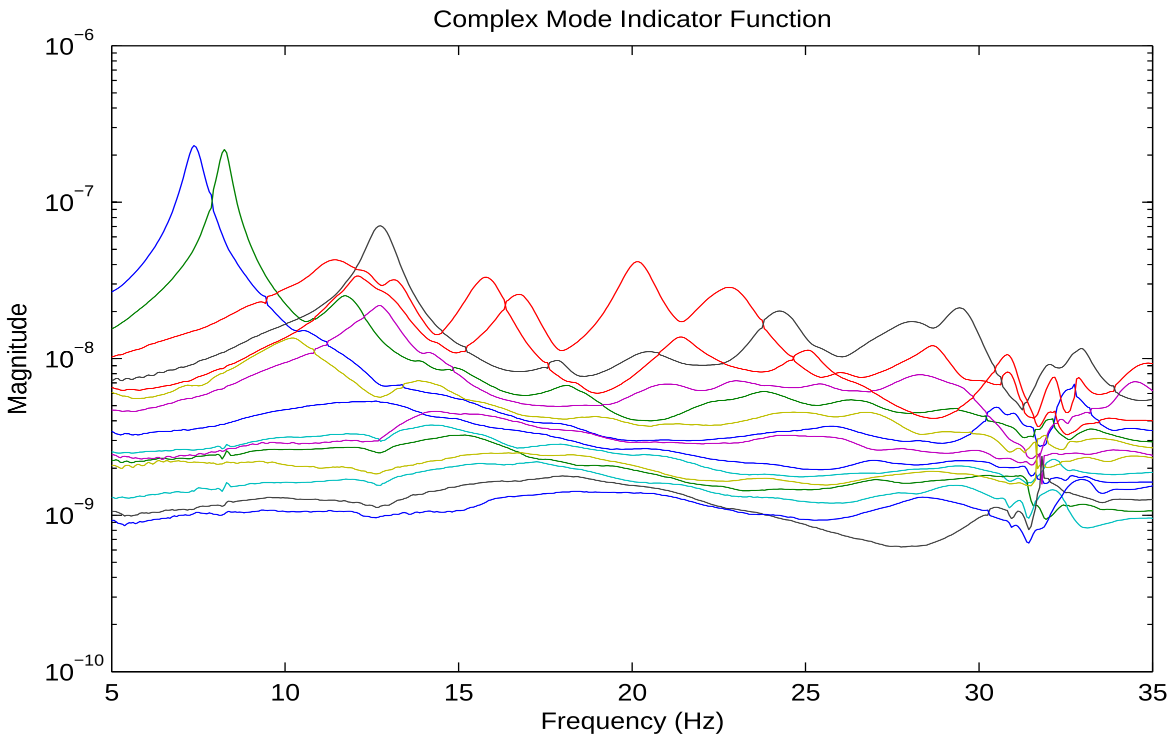

Figure 18. CMIF, Bridge Data

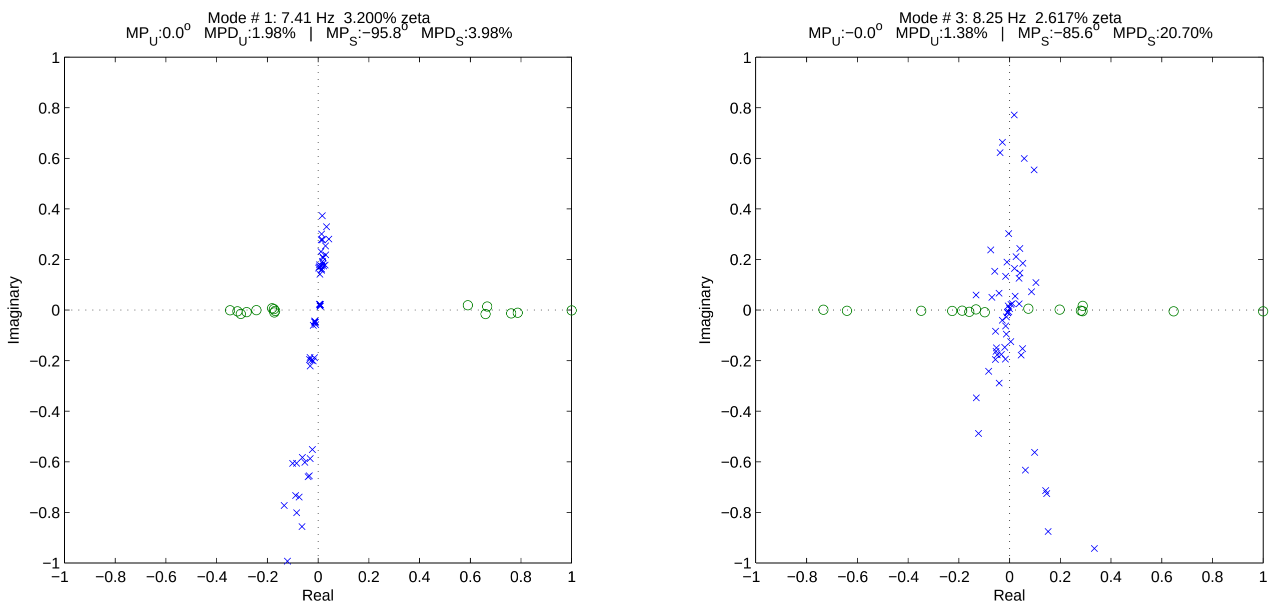

Figure 19. Vector Plots, Bridge Data, RFPZ (Method 2), Mode 1 and Mode 2

Figures 18 and 19 are taken from a civil infrastructure and represent a complicated, real-world system. The dataset consists of 55 inputs (long dimension) and 15 outputs (short dimension). As can be seen in the CMIF plot, there are numerous modes and in some frequency bands there is high modal density which includes both moderate and heavy damping. Also, Figure 15 shows that even though the short(er) dimension, unscaled modal vectors may be calculated to be nearly normal, when computing the full length, long dimension, scaled vectors, some modes are computed as nearly normal, while other modes are clearly significantly complex.

5. Conclusions

This paper has explored the qualitative impact of several estimation procedures on the resulting scaled modal parameters (scaled modal vectors and modal scaling). The four traditional techniques (Methods 1-4) have been compared with a new approach (Method 5) which computes the scaled modal parameters concurrent with the first phase modal frequencies and unscaled modal vectors, while utilizing the first phase computational modes as the residuals. This approach has been shown to be advantageous in providing the analyst with full shape (long dimension) information for traditionally short dimension algorithms (eg. PTD, RFP, et.al.) and complete scaling information for all algorithms. By having such information available, the analyst can make more informed modal parameter selection decisions. The quality of the scaled modal vectors and modal scaling, calculated by this new procedure (Method 5), have been found to be comparable to those computed by traditional approaches (Methods 1-4) and, in most cases, render a secondary, independent estimation of the scaled modal vectors and modal scaling unnecessary. The authors believe that his may be very useful when applied to autonomic modal parameter estimation methods (wizard solutions) that are of great interest for modern, commercial modal parameter implementations.

6. References

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