Impedance Modeling with Multiple Axis Load Cells

Daniel R. Lazor
University of Cincinnati
Structural Dynamics Research Laboratory
P.O. Box 210072
Cincinnati, Ohio 45221-0072 USA

David L. Brown
University of Cincinnati
Structural Dynamics Research Laboratory
P.O. Box 210072
Cincinnati, Ohio 45221-0072 USA

Abstract
Current methods of modeling structures generally have accuracy limitations associated with either frequency range or measurement practicality. Modal modeling can be useful at lower frequencies where modal interactions are relatively few. Impedance modeling requires accurate impedance measurements at the component connection points. Using multiple input multiple output measurements and resolving the inputs and outputs to a common location, a six degree of freedom (DOF) impedance frequency response function (FRF) can be determined. Since the six DOF FRF measurements are determined using computational methods, errors can be introduced into the results. By measuring all six input forces, three translational and three rotational, a more accurate impedance function can be determined. In order to accomplish this, a six DOF load cell was developed to measure the six excitation force components exciting a structure. Presented is an overview of impedance modeling theory and relevant issues arising from the determination of the six degree of freedom impedance functions. An experimental example is given to demonstrate the important theoretical and impedance measurement issues.

Nomenclature

\[
[H] \quad \text{Frequency response functions}
\]

\[
[\Psi^T_x] \quad \text{Response transformation from point } p \text{ to } q
\]

\[
[\Psi^T_F] \quad \text{Reference transformation from point } p \text{ to } q
\]

\[
\{X\} \quad \text{Response vector}
\]

\[
\{F\} \quad \text{Input force vector}
\]

\[
[S] \quad \text{Six DOF load cell electrical outputs}
\]

\[
[H_{SF}] \quad \text{Six DOF load cell sensitivity matrix}
\]

1.0 Impedance Modeling Theory
In structural dynamics, impedance is defined as the ratio between the output velocity and the force input as a function of frequency. The general equation can be defined by

\[
[H]_{N_{x_{i}}N_{i}} \{F\}_{N_{x_{i}}} = \{\dot{X}\}_{N_{x_{i}}} \quad (1)
\]

By simply integrating or differentiating the preceding equation, it can be seen that the impedance relationship is valid for both output displacement and output acceleration with respect to input force. Since impedance functions usually contain contribution of all modes of vibration of a structure, the compact form of the FRFs provide for an efficient start to a structural dynamics model while utilizing a minimum set of physical DOFs.

The motivation for impedance modeling not only lies in the reduction of physical degrees of freedom, but also in the ease of combing sub-structures. By obtaining the complete set of impedance functions at all...
connection points, several sub-structures can be assembled into one complete structure. The result of the model will contain the FRFs of the combined system at all the connection points. Figure 1 shows an analytical example used to demonstrate the ease of combining sub-structures to form a single, combined system from impedance functions.

Two general structural models exist that can be utilized in the formulation of an impedance model. The dynamic stiffness model is defined by

\[ \{ F(\omega) \}_{N_{x1}} = \left[ K(\omega) \right]_{N_{x0}} \{ X(\omega) \}_{N_{x1}} \]  

(2)

The system impedance matrix \([K]\) is a function on the frequency, \(\omega\). In order to obtain the system frequency response functions, \([K]\) must be invertible. In general, this implies that the number of inputs must be equal to the number of outputs. However, a pseudoinverse can be applied, but the condition number must be verified so that the matrix is invertible and nonsingular.

The dynamic compliance model utilizes the inverse stiffness matrix \([H]\).

\[ \{ X(\omega) \}_{N_{x1}} = \left[ H(\omega) \right]_{N_{x0N_{x}}} \{ F(\omega) \}_{N_{x1}} \]  

(3)

This model allows for the difference in the number of inputs and outputs of a system which, most often, tends to be the case in experimental structural dynamic analysis. Since the quantity \([H]\) is typically a measured quantity, it is convenient to use this formulation for impedance modeling applications. Compliance generally refers to outputs in measures of displacement. As previously stated, other quantities can be substituted freely for the output quantity. Thus the dynamic compliance model can be expressed in other terms. From this point, \(X(\omega)\) will refer to the output acceleration of the structure.

1.1 MODEL DEVELOPMENT

For the example shown in Figure 1, there are three sub-systems, A, B and C, that are to be combined using impedance modeling to determine the dynamics of the combined system. Using the compliance model, four equations can be stated to define the systems.

Note that the frequency, \(\omega\), has been removed from the equations for convenience.

\[
\begin{align*}
\{X\}_A^{(4)} &= \left[H\right]_A^{(4)} \{F\}_A^{(4)} \\
\{X\}_B^{(4)} &= \left[H\right]_B^{(4)} \{F\}_B^{(4)} \\
\{X\}_C^{(4)} &= \left[H\right]_C^{(4)} \{F\}_C^{(4)}
\end{align*}
\]

(4)

Writing the equations in similar form yield

\[
\begin{bmatrix}
I & -H^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & -H^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & -H^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & -H^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\{X\}_A^{(4)} \\
\{X\}_B^{(4)} \\
\{X\}_C^{(4)} \\
\{X\}_D^{(4)} \\
\{X\}_E^{(4)} \\
\{X\}_F^{(4)} \\
\{X\}_G^{(4)} \\
\{X\}_H^{(4)}
\end{bmatrix}
= \begin{bmatrix}
\{F\}_A^{(4)} \\
\{F\}_B^{(4)} \\
\{F\}_C^{(4)} \\
\{F\}_D^{(4)} \\
\{F\}_E^{(4)} \\
\{F\}_F^{(4)} \\
\{F\}_G^{(4)} \\
\{F\}_H^{(4)}
\end{bmatrix}
\]

(5)

It can be seen that no coupling exists between the three systems, as would be expected. Impedance modeling can be used to couple the degrees of freedom of sub-systems to other sub-systems. The results of the coupling, in fact combine the sub-systems to form a single, connected system. In order to mechanically couple the systems, compatibility constraints are applied. In general, infinite stiffness without intermediate mass interactions can be assumed across the connection points, thus providing two constraint equations for each connection point DOF. The kinematic constraint from Newton’s First Law states that the forces exerted by two rigidly connected bodies in static equilibrium must be equal and opposite in sign. The kinematic constraint states that two rigidly connected bodies must remain connected, thus there is no relative displacement between the connected bodies.

The compatibility constraints for each connection point can be defined as

\[
\begin{align*}
\{X\}_A^{(4)} &= \{X\}_B^{(1)} & \{F\}_A^{(4)} &= -\{F\}_B^{(1)} \\
\{X\}_C^{(4)} &= \{X\}_B^{(2)} & \{F\}_C^{(4)} &= -\{F\}_B^{(2)}
\end{align*}
\]

(6)

Applying the constraint equations to the original equation set yields

\[
\begin{bmatrix}
I & -H^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & -H^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & -H^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & -H^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\{X\}_A^{(4)} \\
\{X\}_B^{(4)} \\
\{X\}_C^{(4)} \\
\{X\}_D^{(4)} \\
\{X\}_E^{(4)} \\
\{X\}_F^{(4)} \\
\{X\}_G^{(4)} \\
\{X\}_H^{(4)}
\end{bmatrix}
= \begin{bmatrix}
\{F\}_A^{(4)} \\
\{F\}_B^{(4)} \\
\{F\}_C^{(4)} \\
\{F\}_D^{(4)} \\
\{F\}_E^{(4)} \\
\{F\}_F^{(4)} \\
\{F\}_G^{(4)} \\
\{F\}_H^{(4)}
\end{bmatrix}
\]

(7)

This coupled set of equations cannot be used to determine the solution to the model of the connected
system directly. In order to do so, an analytical, external force must be applied to the system. The external force can be applied to any physical DOF with a measured FRF describing the input-output relationship at that DOF.

Since, FRFs can be thought of as the outputs of the system normalized by the inputs to the system. By setting the external force to unity, the system outputs of the solution to the impedance model will, in fact, be the FRFs of the combined system.

For example, if the input DOFs of interest for the combined system are to be determined at the connection point located on sub-system A, the complete set of equations defining the impedance model will be

$$ [I] - [H]^e_{c} \begin{bmatrix} [0] & [0] & [0] & [0] & [0] & [0] & -[H]^e_{c} \end{bmatrix} \begin{bmatrix} [X]^e \end{bmatrix} = \begin{bmatrix} [0]^e \end{bmatrix} $$

Only one external force can be used to solve the preceding equation at a time. Therefore, by substituting each column of sub-systems A’s impedance function matrix into $[H]^e_{c}$ in Equation 8 will yield the FRFs of the combined system due to an excitation at that point. When the impedance function for sub-system A is a 6x6 matrix, Equation 8 must be solved six times in order to obtain the combined systems FRFs due to the all six external input forces applied at the connection point.

2.0 INDIRECT MEASUREMENTS

In the derivation given for the impedance model, the impedance functions used at the connection point were given using all six degrees of freedom. Not only are the translational accelerations and forces considered, but the angular accelerations and moments are used as well. In actuality, an impedance model can be computed utilizing only a single DOF connection point. However, the problem introduced by using a minimal set of DOFs is a reduction in the accuracy of the model. Whenever there is a high degree of coupling between DOFs and one of these DOFs is neglected in the modeling process, large errors can result in the solution. Thus it is crucial to measure all degrees of freedom between connection points to obtain the highest degree of model accuracy when system coupling is unknown.

In measurements, the development of tri-axial accelerometers and tri-axial load cells has increased the accuracy of impedance models, but few transducers exist that can be utilized to measure angular accelerations and moments for structural dynamics applications. Therefore the indirect method for measuring rotations can be applied.\[1\]

Rigid body dynamics are used to approximate the translational and rotational measurement DOFs at a point on a rigid structure due to a redundant set of measurements elsewhere on the rigid body structure. In general, a linear force applied at one point on a rigid body can be represented by an equivalent force-moment couple elsewhere on the rigid body using a transformation matrix.

$$ \begin{bmatrix} F_x \ F_y \ F_z \end{bmatrix} = [\Psi]^q_{F} \begin{bmatrix} F_x^p \ F_y^p \ F_z^p \end{bmatrix} $$

The transformation matrix used to describe an equivalent force-moment couple at point q for a six degree of freedom force-moment input at point p is defined as

$$ [\Psi]^q_{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ z_q - z_p & (y_q - y_p) & 0 & 0 & 0 & 0 \\ (z_p - z_q) & 0 & (x_q - x_p) & 0 & 1 & 0 \\ (y_q - y_p) & (x_q - x_p) & 0 & 0 & 0 & 1 \end{bmatrix} $$

Typically, for input excitations from a modal tuned impact hammer or electro-dynamic shaker the force transformation matrix can be truncate to a 6x3 matrix since no moment are measured at the input location.

Similarly, the accelerations measured at one point on a rigid body can be represented by an equivalent translational-rotational acceleration. The difference lies in the fact that the transformed accelerations are the superposition of all rigid body modes of the structure.

$$ [\Psi]^q_{X} = \begin{bmatrix} X^q \ Y^q \ Z^q \end{bmatrix} = [\Psi]^p_{X} \begin{bmatrix} X^p \ Y^p \ Z^p \end{bmatrix} $$

The transformation matrix used to describe the motion at point q due to the summation of rigid body modes at point p is defined as.
Since six degree of freedom accelerations are not typically measured, the last three rows can be omitted. Thus, equation 12 becomes

\[
\begin{bmatrix}
1 & 0 & 0 & (z_p - z_q) & (y_q - y_p) \\
0 & 1 & 0 & (z_q - z_p) & 0 \\
0 & 0 & 1 & (y_p - y_q) & (x_q - x_p)
\end{bmatrix}
\] (13)

In order to determine the motion at point \( q \) on the rigid body, the transformation matrix is moved to the other side of the equation.

\[
\begin{bmatrix}
1 & 0 & 0 & (z_p - z_q) & (y_q - y_p) \\
0 & 1 & 0 & (z_q - z_p) & 0 \\
0 & 0 & 1 & (y_p - y_q) & (x_q - x_p)
\end{bmatrix} = \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\] (14)

This implies that the transformation matrix is invertible. To accommodate this requirement, redundant perimeter measurements on the rigid body must be made. A minimum of three measurements in each of the three translational directions needs to be made to exactly solve for the motion at point \( q \). By increasing the number of measurements over the required number, a least squares solution can be determined.

The equations shown above have been derived for a specific case of sensor orientation. When the excitation line of action or the accelerometers are inline with the global system coordinate space, the above equations result. However, in the case of a skewed impact or a sensor that is mounted on a curved surface, the line of action and sensor orientation will not be in the global system coordinate space. For these situations, the equations can be modified to include the direction cosines that relate the sensor and line of action coordinate systems to the global system coordinates. By pre-multiplying the measured tri-axial accelerometer measurements by the direction cosine matrix, the responses can be transformed into the global system coordinate. Also, the direction cosines can be used to represent the three force components equivalent to skewed excitation orientation.\(^2\)

2.1 INDIRECT FRF MEASUREMENTS

The indirect measurement method can be directly applied to an FRF matrix in order to compute a six degree of freedom impedance measurement at a point on a rigid structure. For the purpose of indirect measurements, a rigid structure is any part of the structure that is used for indirect measurement synthesis and does not undergo motion due to deformation modes in the frequency range of interest.

By combining Equations 3, 12, and 13 the solution for the six degree of freedom impedance FRFs can be reduced to

\[
[H]_{6 \times 6}^q = [\Psi_X]_{6 \times 6}^+ [H]_{NaxNf} [\Psi_F]_{Nix6}^T (15)
\]

In order to obtain an accurate six DOF impedance function, several references and responses need to be applied and measured. Inputs into the system must be numerous enough to provide adequate excitation in all translational and rotational directions. A multiple reference impact test is well suited for use with indirect measurements, since a large amount of impacts can be obtained in a relatively short amount of test time. Also, by using several tri-axial accelerometers, an impedance FRF matrix for a sub-structure can be computed and is ideal for impedance modeling.

Figure 2 is an example of the indirect measurement application. The 22 single arrows represent input excitations to the system. Tri-axial accelerometers are located at the eight corners of the plate providing for 24 translational acceleration measurements. By applying Equation 12 to the FRF matrix with 24x22 basis, an equivalent 6x6 impedance matrix can be computed anywhere on the plate.

Figure 2: Indirect measurements application.

Since impedance modeling only requires measurements at the connection degrees of freedom, the only physical measurements required are those on the rigid body used to compute the six DOF impedance functions using the indirect method. Thus, an impedance model can be achieved with fewer measurements than would be required to perform a modal model on the same structure.
3.0 MULTI-AXIS LOAD CELL CALIBRATION

A load cell has been developed at the University of Cincinnati Structural Dynamics Research Laboratory which can be used to measure all six force components acting through the sensor. The sensor consists of an array of twelve strain gages mounted to connecting arms of the sensor. Since there are 12 voltage outputs from the sensor, it is reasonable to assume that the strain gages are mechanically coupled to other strain gages. Therefore a single calibration value cannot be used for each sensor in order to compute the forces acting through the load cell. A calibration method was developed to create a calibration matrix that converts the electrical outputs of the sensor to the six component forces.\[3\]

The calibration procedure uses a mass with known inertia properties that acts as a rigid body in the calibration frequency range. The indirect measurement method is utilized to determine translational and rotational accelerations about the connection point between the load cell and the calibration mass.

\[
{\{F_6\}^y_{\text{NixNa}}} = \begin{bmatrix} m \end{bmatrix}_{6\times6} \begin{bmatrix} \Psi \end{bmatrix}_{6\times6} \begin{bmatrix} \ddot{X} \end{bmatrix}_{\text{NixNa}}
\]

(16)

The preceding equation show that by combining the indirect measurement method with Newton’s Second Law, an estimation of the forces that exist at the connection point between the load cell and the calibration mass can be estimated. In order to determine a sensitivity matrix for the load cell, the ratio of the estimated forces and the electrical outputs measured from the multi-axis load cell can be determined.

\[
{\{H_{SF}\}^y_{\text{Nix6}}} = \begin{bmatrix} [S] \end{bmatrix}_{\text{NixNa}}
\]

(17)

Using an H₁ FRF estimation algorithm, the sensitivity matrix of the multi-axis load cell can be computed.

\[
{\{H_{SF}\}^y_{\text{Nix6}}} = \begin{bmatrix} [G_{SF}] \end{bmatrix}_{\text{Nix6}} \begin{bmatrix} [G_{FF}] \end{bmatrix}_{6\times6}
\]

(18)

This sensitivity matrix can now be applied to the electrical outputs of the multi-axis load cell to determine the forces acting through the load cell.

\[
{\{F_6\}^y_{\text{Nix1}}} = \begin{bmatrix} \{S \} \end{bmatrix}_{1\times6} \begin{bmatrix} \{H_{SF}\}^y_{\text{Nix6}} \end{bmatrix}
\]

(19)

Since the sensitivity matrix is applied by utilizing a pseudoinverse approach, the matrix must be invertible. By verifying that the condition number of this matrix is reasonably small, this method can be applied.

4.0 EXPERIMENTAL EXAMPLE

In order to validate the procedures previously mentioned, an experimental structure was instrumented and tested. The results of each test were compared with the measured solution in order to determine the accuracy of the measurements and models.

4.1 EXPERIMENTAL H-FRAME

The test article used to validate the measurements was an h-frame made from structural steel box tubing and aluminum endplates with a total mass of 110 kg. The h-frame is designed to be relatively uncoupled in some directions while possessing a high degree of coupling in others. The structure is supported by light bungee cords in order to approximate a free-free boundary condition.
The h-frame was tested in three configurations. The unmodified structure was tested to provide a baseline set of measurements. Another test was taken with a single piece of steel bolted to the endplates. The last test had two additional pieces of steel added to the single steel plate. The masses were considered to be sub-structures for use with the impedance model.

The aluminum endplates on each end of the h-frame legs act as rigid bodies at lower frequencies. The first deformation mode of the endplates is at 918 Hz, thus the indirect measurement method can be utilized to compute a six degree of freedom impedance measurement on the endplate below the deformation mode. This is convenient since, the mass additions are connected to one of the h-frame endplates.

Since the added structures have simple geometries, the FRFs for the added mass were compute from the mass inertia matrix. The plate sub-structures were modeled in AutoCAD, and the inertia matrices were easily obtained from the mass of the plates, center of gravities, and the inertia properties.

The first test scenario obtained a baseline FRF impedance matrix for the unmodified h-frame. This was accomplished by performing a multiple reference impact test and synthesizing the impedance matrix at the center of the endplate using the indirect measurement method. Figures 7 and 8 show reciprocity measurements for the six DOF impedance matrix. Figure 7 shows good correlation between the X-axis translation and Y-axis rotation which indicates that these measurement directions exhibit a great deal of modal coupling. However, it can be seen that in Figure 8 reciprocity fails due to the lack of modal coupling between the X-axis and Y-axis translations.

At the anti-resonances, the FRFs differ slightly due to the sensitivity to impact orientation and location. Slight misalignments and deviations in impact location can have a large effect on the anti-resonances when computing a 6 DOF impedance function from the indirect measurement method.

The second test scenario utilized the multi-axis force sensor to determine the six DOF forces acting on the test structure. To ensure that all degrees of freedom have been adequately excited and to determine how well the data fits the linear model for the multi-axis load cell data, a principle component analysis and a multiple coherence calculation were performed.
Figure 9 shows that the excitation to the system through the load cell provided a good signal to noise ratio, and that all translational measurements were at the same level as well as rotational measurements occurring at the same level. The fact that there is an offset between the translational and rotational forces is not significant since the measurements can be scaled by changing the displacement units on the calibration data.

For the unmodified h-frame dataset using the multi-axis load cell, Figure 10 shows high multiple coherence values for a majority of the frequency range of interest. It is reasonable to assume that the six DOF accelerations correlate well with the input excitations measured by the load cell.

4.2 IMPEDANCE MODELING RESULTS

The first impedance model case used a single steel plate weighing 15.2 kg. To verify results of the modeling, the impedance functions obtained from the baseline data using the indirect measurement method and the impedance functions obtained from the multi-axis load cell were analytically connected to the steel plate using impedance modeling. These results were then compared to the impedance functions obtained from physically connecting the steel plate to the endplate on the h-frame and applying the indirect measurement method to the acquired dataset.

Figure 11: Mass added on h-frame endplates.

Figure 12 shows the comparison of the driving point impedance model results with the data from the physical system. The X-axis and Y-axis translations along with the Z-axis rotation show good correlation between the predicted results and the measured results. The discrepancies in the Z-axis translation can be explained by the physical makeup of the h-frame. There exists a high degree of stiffness in the Z-axis direction which makes obtaining accurate measurements extremely difficult. This direction is highly sensitive to impact hammer misalignments and variances in impact locations. Since all the measurements use the indirect method to compute the angular accelerations at the connection point, errors are introduced due to any Z-axis accelerometer measurement. Equation 14 shows that the X-axis and Y-axis rotations are directly affected by errors associated with Z-axis translational measurement errors, thus explaining the deviation from the measured values for these degrees of freedom.

At higher frequencies, the model mismatch becomes greater. Since data was only taken in the frequency range of 0 to 800 Hz, modes that exist higher than 800 Hz are being pulled down into the measured frequency range by the additional of the large mass. The data used in the model cannot be used to predict these effects since no information exists about these higher frequency modes.

With the addition of the two smaller steel plates to the single steel plate, the total mass added to the system increases to 27.5 kg or a quarter of the original weight of the h-frame. Figure 13 shows the comparison between the model results and the measured data for the heavy mass addition. The results are similar to that of the single plate comparison, but the errors are more pronounced.
The mass addition to the system presents a rather large change in the dynamics of the h-frame. Thus errors resulting from impacts misalignments, impact location deviations, and accelerometer orientation and location errors will be amplified. Figure 14 demonstrates the dynamic effects resulting from the addition of the masses to the h-frame. The modal frequencies for the single plate addition lower typically by 15% and 25% for the heavier mass addition. The Z-rotation measures drastically change from 290 Hz to 137.5 Hz and 122.5 Hz for the lighter and heavier mass additions respectively.

In order to accurately measure a complete 6 DOF impedance measurement at a connection point for a complex structure, all 6 DOFs for forces and accelerations is necessary to eliminate the effects of deformation modes for indirect measurements. Methods exist to filter the effects of deformation modes, but become impractical when a large amount of deformation modes exist in a relatively small frequency range.

6.0 ACKNOWLEDGEMENTS

The authors would like to thank The Modal Shop and PCB Piezotronics for providing equipment essential to performing the measurements. Also, the authors would like to thank Jeff Hylok of the UC-SDRL for assisting in data acquisition and Susan DeClerq for the work on the multi-axis load cell.

7.0 REFERENCES


