A Low Order Frequency Domain Algorithm for Operational Modal Analysis

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Abstract

Most of the algorithms for Operational Modal Analysis work in the time domain and there are very few frequency domain based algorithms. One of the reasons for this is the poor numerical characteristics associated with higher order frequency domain algorithms like Rational Fraction Polynomial (RFP). These limitations can be improved by using methods such as frequency normalization and use of orthogonal polynomials in the traditional experimental modal analysis set up. However estimating modal parameters in the frequency domain using output-only response data still remains a challenge. In this paper a low order frequency domain algorithms is proposed for Operational Modal Analysis. This algorithm is derived using the Unified Matrix Polynomial Approach (UMPA) and is evaluated using a theoretical and an experimental study.

1 Introduction

One of the earliest Operational Modal Analysis (OMA) algorithms to be developed was the Natural Excitation Technique (NExT) [1] which utilized the cross-correlation functions between the measured output responses. Soon many of the traditional Experimental Modal Analysis (EMA) time domain algorithms such as Ibrahim Time Domain (ITD) [2-3], Eigensystem Realization Algorithm (ERA) [4-5], Poly-reference Time Domain (PTD) [6-7], etc were reformulated to suit the operational modal analysis framework. These algorithms also used correlation functions for modal parameter estimation. State space model based modal parameter algorithms like Stochastic Subspace Identification (SSI) [8] algorithm were also developed that worked in the time domain.

Brincker, Zhang and Andersen [9] proposed an algorithm named Frequency Domain Decomposition (FDD) that involved frequency by frequency singular value decomposition of the output power spectra. This algorithm, in principle, was similar to conventional CMIF algorithm [10] which is essentially a spatial domain algorithm rather than a typical frequency domain algorithm. To completely identify a particular mode, the FDD algorithm is followed by enhanced Frequency Domain Decomposition (eFDD) [11] algorithm that determines the damping associated with the mode which wasn’t possible with FDD. The modal parameter estimation portion of the eFDD algorithm again worked in the time domain.

Thus modal parameter identification in the case of OMA is mostly restricted to time domain algorithms. One of the reasons for the lack of frequency domain algorithms for OMA is the poor numerical conditioning associated with the high order frequency domain algorithms such as Rational Fraction Polynomial (RFP) [12]. This problem is much more severe in the case of OMA as the order of the power spectrum based model used in OMA is twice that of the frequency response function based model used in EMA. Recently a method called PolyMAX [13-15] that builds upon the classical least squares complex frequency domain estimator was proposed and shown to work in an OMA framework.

The work presented in this paper involves the utilization of the Unified Matrix Polynomial Approach (UMPA) [16, 17] for developing a low order frequency domain algorithm (UMPA-LOFD) suited for the
output response based OMA framework. The algorithm is applied to an analytical system and also a lightly damped circular plate. It is shown to have better numerical characteristics and results are comparable to time domain based OMA algorithms.

2 Theoretical Background

2.1 Unified Matrix Polynomial Approach (UMPA)

This section describes briefly the theory behind the Unified Matrix Polynomial Approach and its utilization for developing the low order frequency domain algorithm for OMA. To understand the UMPA formulation, the polynomial model used for frequency response functions is considered.

\[
H_{pq}(\omega) = \frac{X_p(\omega)}{F_q(\omega)} = \sum_{k=0}^{n} \beta_k (j \omega)^k
\]

Rewriting this model for a general multiple input, multiple output case and stating it in terms of frequency response functions

\[
\left[ \sum_{k=0}^{m} (j \omega)^k [\alpha_k] \right] [H(\omega)] = \left[ \sum_{k=0}^{n} (j \omega)^k [\beta_k] \right]
\]

This model in the frequency domain is the AutoRegressive with eXogenous inputs (ARX(m,n)) model that corresponds to the AutoRegressive (AR) model in time domain for the case of free decay or impulse response data

\[
\sum_{k=0}^{n} [\alpha_k] [h_{pq}(t_{r,k})] = 0
\]

The general matrix polynomial model concept recognizes that both time and frequency domain models generate functionally similar matrix polynomial models. This model which describes both domains is thus termed as Unified Matrix Polynomial Approach (UMPA) [16, 17].

2.2 Low Order Frequency Domain Algorithm

2.2.1 General UMPA Formulation

Lower order, frequency domain algorithms are basically UMPA based models that generate first or second order matrix coefficient polynomials [18, 19]. Starting with the multiple input, multiple output frequency response model of Eq. (2) and forming a second order matrix polynomial model.

\[
\left[ \sum_{k=0}^{m} (j \omega)^k [\alpha_k] \right] [H(\omega)] = \left[ \sum_{k=0}^{n} (j \omega)^k [\beta_k] \right]
\]

for order m = 2

\[
[\alpha_2 (j \omega)^2 + [\alpha_1 (j \omega) + [\alpha_0] \left[ H(\omega) \right] = [\beta_1 (j \omega) + [\beta_0]
\]

(4)
This basic equation can be repeated for several frequencies and the matrix polynomial coefficients can be obtained using either $[\alpha_2]$ or $[\alpha_0]$ normalization.

$[\alpha_2]$ Normalization

$$
\begin{bmatrix}
[\alpha_0] & [\alpha_1] & [\beta_0] & [\beta_1]
\end{bmatrix}_{N_b \times 4N_i}
\begin{bmatrix}
(j\omega)^2[H(\omega)] & (j\omega)^2[H(\omega)] \\
(j\omega)^2[H(\omega)] & -(j\omega)^2[I] \\
-(j\omega)^2[I] & -(j\omega)^2[I]
\end{bmatrix}_{4N_b \times N_i} = -(j\omega)^2[H(\omega)]_{N_b \times N_i}
$$

$[\alpha_0]$ Normalization

$$
\begin{bmatrix}
[\alpha_1] & [\beta_0] & [\beta_1]
\end{bmatrix}_{N_b \times 4N_i}
\begin{bmatrix}
(j\omega)^2[H(\omega)] & (j\omega)^2[H(\omega)] \\
(j\omega)^2[H(\omega)] & -(j\omega)^2[I] \\
-(j\omega)^2[I] & -(j\omega)^2[I]
\end{bmatrix}_{4N_b \times N_i} = -(j\omega)^2[H(\omega)]_{N_b \times N_i}
$$

The coefficients are then used to form a companion matrix and eigenvalue decomposition can be applied to estimate the modal parameters.

### 2.2.2 Extending UMPA to Operational Modal Analysis

The formulation as described above can be utilized to suit the operational modal analysis framework by using power spectrums instead of frequency response functions. If $\{X(\omega)\}$ is the measured response and $\{F(\omega)\}$ is the input force, the relationship between them in terms of frequency response function $[H(\omega)]$ is given as follows [20]:

$$
\{X(\omega)\} = [H(\omega)]\{F(\omega)\}
$$

(7)

$$
\{X(\omega)\}^\text{H} = \{F(\omega)\}^\text{H} [H(\omega)]^\text{H}
$$

(8)

Now multiplying Eq. (7) and Eq. (8)

$$
\{X(\omega)\}\{X(\omega)\}^\text{H} = [H(\omega)]\{F(\omega)\}^\text{H} [F(\omega)]^\text{H} [H(\omega)]^\text{H}
$$

or with averaging,

$$
[G_{XX}(\omega)] = [H(\omega)] [G_{FF}(\omega)] [H(\omega)]^\text{H}
$$

(9)

where $[G_{XX}(\omega)]$ is the output response power spectra and $[G_{FF}(\omega)]$ is the input force power spectra.

It should be noted that in OMA the power spectra of the input force is assumed to be broadband and smooth. This means that the input power spectra is constant and has no poles or zeroes in the frequency range of interest. The forcing is further assumed to be uniformly distributed spatially ($N_i$ approaching $N_o$, considering the response is being measured all over the structure). Thus the output response power spectra $[G_{XX}(\omega)]$ is proportional to the product $[H(\omega)]^2$ and the order of output response power spectrum is twice that of frequency response functions. This means the power spectrum based UMPA model will be twice the order of a frequency response function based UMPA model.

Since $[G_{FF}(\omega)]$ is constant, $[G_{XX}(\omega)]$ can be expressed in terms of frequency response functions using the polynomial model in Eq. (1).
\[ G_{xx}(\omega) = \left[ \sum_{k=0}^{n} \left( \frac{\beta_k j^k}{\sum_{k=0}^{m} (\alpha_k j^k)} \right) \right] \left[ \sum_{k=0}^{n} \left( \frac{\beta_k j^k}{\sum_{k=0}^{m} (\alpha_k j^k)} \right)^\dagger \right] \]

Since \((n < m)\), a partial fraction model can be formed for the output power spectrum. This partial fraction model for a particular response location \(p\) and reference location \(q\) is given by

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R^*_{pqk}}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - \lambda_k} + \frac{S^*_{pqk}}{j\omega - \lambda_k^*}
\]

(12-a)

and can be more conveniently written as [15]

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R^*_{pqk}}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - \lambda_k} + \frac{S^*_{pqk}}{j\omega - \lambda_k^*}
\]

(12-b)

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R^*_{pqk}}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - (\lambda_k^*)} + \frac{S^*_{pqk}}{j\omega - (\lambda_k^*)}
\]

(12-c)

where \(S_{pqk}\) and \(S^*_{pqk}\) are redefined to incorporate \((-1)\).

Note that \(\lambda_k\) is the pole and \(R_{pqk}\) and \(S_{pqk}\) are the \(k^{th}\) mathematical residues. These residues are different from the residue obtained using a frequency response function based partial fraction model since they do not contain modal scaling factor (as no force is measured). The form of Eq. (12-c) clearly indicates that the roots that will be found from the power spectrum data will be \(\lambda_k, \lambda_k^*, -\lambda_k\) and \(-\lambda_k^*\) for each model order 1 to \(N\).

The high order of the power spectrum based model in comparison to FRF based model causes various disadvantages like numerical conditioning, effect of noise etc which makes it more difficult for the frequency domain based algorithm to give good results as they inherently suffer from numerical conditioning problems. However this problem can be tackled by using positive power spectra [21]. In this approach first power spectrums are calculated from measured output time responses and then respective correlation functions are obtained by inverse Fourier transformation. After this step, the negative lag portion of the correlation functions is set out to zero and the positive power spectra are obtained by Fourier transforming this resultant data. This is mathematically represented as

\[
G_{pq}^+(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R^*_{pqk}}{j\omega - \lambda_k^*}
\]

(13)

It is clear from this equation that not only the order of positive power spectrum is the same as that of the frequency response functions, but also they contain all the necessary system information. Now Eq. 4-6 can be formed based on \([G_{xx}(\omega)]^+\) data and second order UMPA model can be utilized for the purpose of modal parameter estimation.
3 Experimental Studies

3.1 Analytical 15 Degree of Freedom System

The low order frequency domain algorithm UMPA-LOFD is applied to an analytical 15 degree of freedom system as shown in Figure 1. The system is excited by a white random uncorrelated input at all 15 degrees of freedom. Output response power spectrums are obtained from the raw time data using the correlogram method [22, 23] and these power spectrums are further processed to obtain positive power spectrums as explained in previous section.

Figure 1: Analytical 15 Degree of Freedom System

Figure 2 shows the auto power spectrum and positive power spectrum for the degree of freedom number 1 or driving point 1 (G_{XX_{11}} and G_{X_{X11}}). A complex mode indicator function (CMIF) plot based on power spectrums as shown in Figure 3 indicates clearly the presence of all 15 modes including a repeated mode around 53.3 Hz.

Figure 2: Auto power spectrum and positive power spectrum for the first degree of freedom
Modal parameters obtained from UMPA-LOFD algorithm are listed in Table along with those obtained from time domain algorithms like ERA, PTD. It should be noted that though these algorithms are referred by the name through which they are known popularly in the conventional frequency response function based experimental modal analysis framework, in this study they are essentially operational modal analysis algorithms i.e. working on output-only data. As can be seen, the results of the UMPA-LOFD algorithm compare very well with the time domain algorithms. Though the damping is slightly off (This is the case with time domain algorithms as well), the error is relatively insignificant as damping is given in terms of percentage critical.

Table 1: Analytical System Modal Parameter Comparison

<table>
<thead>
<tr>
<th>True Modes</th>
<th>UMPA-LOFD (Low Order, Frequency Domain)</th>
<th>UMPA-ERA (Low Order, Time Domain)</th>
<th>UMPA-PTD (High Order, Time Domain)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Damp</td>
<td>Freq</td>
<td>Damp</td>
</tr>
<tr>
<td>1.0042</td>
<td>15.985</td>
<td>2.338</td>
<td>15.963</td>
</tr>
<tr>
<td>2.7347</td>
<td>43.6</td>
<td>3.043</td>
<td>43.680</td>
</tr>
<tr>
<td>2.9122</td>
<td>46.444</td>
<td>3.431</td>
<td>46.437</td>
</tr>
<tr>
<td>3.3375</td>
<td>53.317</td>
<td>3.932</td>
<td>53.209</td>
</tr>
<tr>
<td>3.7145</td>
<td>59.413</td>
<td>4.430</td>
<td>59.116</td>
</tr>
<tr>
<td>3.858</td>
<td>61.624</td>
<td>4.180</td>
<td>61.133</td>
</tr>
<tr>
<td>4.2978</td>
<td>68.811</td>
<td>4.291</td>
<td>69.237</td>
</tr>
<tr>
<td>4.5925</td>
<td>73.63</td>
<td>4.812</td>
<td>73.253</td>
</tr>
<tr>
<td>2.6093</td>
<td>128.84</td>
<td>2.712</td>
<td>128.848</td>
</tr>
<tr>
<td>2.4548</td>
<td>136.55</td>
<td>2.563</td>
<td>136.547</td>
</tr>
<tr>
<td>2.3288</td>
<td>143.86</td>
<td>2.426</td>
<td>143.869</td>
</tr>
<tr>
<td>2.221</td>
<td>150.83</td>
<td>2.314</td>
<td>150.799</td>
</tr>
</tbody>
</table>
Figure 4 through Figure 7 show the consistency diagrams obtained using various algorithms (ERA, PTD, UMPA-LOFD and RFP). Except for RFP, the consistency diagrams obtained using other algorithms are very clear and show good stability of the modes. Note that the diamonds (◊) in the consistency diagram represent stable pole and vector. The poor quality of consistency diagram obtained using RFP algorithm (Figure 7) makes it useless to be used for modal parameter estimation process. It is also shown by means of Figure 8 that processing the positive power spectrums rather than normal power spectrums result in much clear and definitive consistency diagrams.

![Consistency Diagram](image)

**Figure 4: Consistency diagram for Polyreference Time Domain (PTD) algorithm (Analytical system)**
Figure 5: Consistency diagram for Eigensystem Realization Algorithm (ERA) (Analytical system)

Figure 6: Consistency diagram for Low Order Frequency Domain (UMPA-LOFD) algorithm (Analytical system)
Figure 7: Consistency diagram for Rational Fraction Polynomial (RFP) algorithm (Analytical system)

Figure 8: Consistency diagram for Low Order Frequency Domain (UMPA-LOFD) algorithm based on complete power spectrum (Analytical system)
3.2 Lightly Damped Circular Plate

A lightly damped circular plate made of aluminum is randomly across the entire surface using an impact hammer. The response is measured by placing accelerometers at 30 locations on the plate (Figure 9). A two shaker test is also conducted to compare the OMA results with EMA results. It should be noted that for the shaker test the plate is excited at two locations by an ergodic, stationary, broad-band pure random signal.

Figure 10 shows the CMIF plot based on power spectrums and clearly indicates the presence of the various modes present.

![Figure 9: Experimental set up for the lightly damped circular plate](image)

![Figure 10: CMIF plot based on complete power spectrums obtained when plate is excited sufficiently over its surface](image)
### Table 2: Lightly Damped Circular Plate Modal Parameter Comparison

<table>
<thead>
<tr>
<th>System Modes Using EMA</th>
<th>UMPA-LOFD (Low Order, Frequency Domain)</th>
<th>UMPA-ERA (Low Order, Time Domain)</th>
<th>UMPA-PTD (High Order, Time Domain)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damp</td>
<td>Freq</td>
<td>Damp</td>
<td>Freq</td>
</tr>
<tr>
<td>0.258</td>
<td>56.591</td>
<td>0.612</td>
<td>56.478</td>
</tr>
<tr>
<td>0.285</td>
<td>57.194</td>
<td>0.621</td>
<td>57.253</td>
</tr>
<tr>
<td>0.312</td>
<td>96.577</td>
<td>0.636</td>
<td>96.665</td>
</tr>
<tr>
<td>0.412</td>
<td>132.101</td>
<td>0.342</td>
<td>131.830</td>
</tr>
<tr>
<td>0.147</td>
<td>132.650</td>
<td>0.285</td>
<td>132.760</td>
</tr>
<tr>
<td>0.243</td>
<td>219.582</td>
<td>0.300</td>
<td>219.375</td>
</tr>
<tr>
<td>0.216</td>
<td>220.952</td>
<td>0.364</td>
<td>221.358</td>
</tr>
<tr>
<td>0.214</td>
<td>231.172</td>
<td>0.256</td>
<td>230.851</td>
</tr>
<tr>
<td>0.137</td>
<td>232.077</td>
<td>0.225</td>
<td>232.394</td>
</tr>
<tr>
<td>0.089</td>
<td>352.997</td>
<td>0.147</td>
<td>351.677</td>
</tr>
<tr>
<td>0.174</td>
<td>355.509</td>
<td>0.219</td>
<td>355.773</td>
</tr>
<tr>
<td>0.180</td>
<td>374.554</td>
<td>0.268</td>
<td>373.933</td>
</tr>
<tr>
<td>0.176</td>
<td>377.569</td>
<td>0.236</td>
<td>377.505</td>
</tr>
<tr>
<td>0.313</td>
<td>412.414</td>
<td>0.241</td>
<td>411.727</td>
</tr>
<tr>
<td>0.209</td>
<td>486.801</td>
<td>0.219</td>
<td>485.405</td>
</tr>
</tbody>
</table>

The modal parameters estimated using various algorithms including the UMPA-LOFD algorithm are listed in Table and show good agreement among them. Figures 11-14 show the consistency diagrams for the different algorithms. The consistency diagram for UMPA-LOFD algorithm is much clearer than the other frequency domain algorithm RFP. It is also comparable to consistency diagrams of the time domain algorithms.
Figure 11: Consistency diagram for Polyreference Time Domain (PTD) algorithm (Circular plate)

Figure 12: Consistency diagram for Eigensystem Realization Algorithm (ERA) algorithm (Circular plate)
Further the independence of the various estimated modes is checked by the means of modal assurance criterion (MAC) plot as shown in Figure 15. It is evident that all the 15 modes, most of which are closely spaced modes, are independent and represent different modes of the system. The mode shapes obtained for
the circular plate are shown in Figure 16. The mode shapes are of similar nature to the ones obtained through experimental modal analysis, except that they are not scaled.
Conclusions

A frequency domain algorithm based on power spectrum data is developed using Unified Matrix Polynomial Approach (UMPA). This algorithm, UMPA-LOFD, is a low order algorithm and offers a frequency domain alternative for operational modal analysis purposes. The algorithm is shown to have comparable results as the time domain algorithms and has better numerical characteristics than the high order frequency domain algorithms like the Rational Fraction Polynomial algorithm. The good results obtained using the algorithm makes it a good addition to the family of operational modal analysis algorithms.

References


