Unified Matrix Polynomial Approach for Operational Modal Analysis

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Nomenclature

\( t, \omega, z, s \) \hspace{1cm} \text{Time, Frequency, Z, Laplace}

\([M]\) \hspace{1cm} \text{Mass Matrix}

\([C]\) \hspace{1cm} \text{Damping Matrix}

\([K]\) \hspace{1cm} \text{Stiffness Matrix}

\( \{X(s)\} \) \hspace{1cm} \text{Response Vector}

\( \{F(s)\} \) \hspace{1cm} \text{Force Vector}

\( H(*) \) \hspace{1cm} \text{Frequency Response Function Matrix}

\( N_o \) \hspace{1cm} \text{Number of Output Degrees of Freedom}

\( N_i \) \hspace{1cm} \text{Number of Input Degrees of Freedom}

\( N_{ref} \) \hspace{1cm} \text{Number of Reference Responses}

\( G_{ff}(*) \) \hspace{1cm} \text{Input Force Power Spectrum}

\( G_{xx}(*) \) \hspace{1cm} \text{Output Response Power Spectrum}

\( R_{xx} \) \hspace{1cm} \text{Correlation Function}

\( G^+ \) \hspace{1cm} \text{Positive power spectrum}

\( \alpha, \beta \) \hspace{1cm} \text{Polynomial Coefficients}

\( R, S \) \hspace{1cm} \text{Residue Matrices}

\( \lambda \) \hspace{1cm} \text{Modal Frequency}

\( \text{LSCE} \) \hspace{1cm} \text{Least Squares Complex Exponential}

\( \text{ERA} \) \hspace{1cm} \text{Eigensystem Realization Algorithm}

\( \text{PTD} \) \hspace{1cm} \text{Polyreference Time Domain}

\( \text{PFD} \) \hspace{1cm} \text{Polyreference Frequency Domain}

\( \text{UMPA} \) \hspace{1cm} \text{Unified Matrix Polynomial Approach}

\( \text{EMA} \) \hspace{1cm} \text{Experimental Modal Analysis}

\( \text{OMA} \) \hspace{1cm} \text{Operational Modal Analysis}

\( \text{CEA} \) \hspace{1cm} \text{Complex Exponential Algorithm}

\( \text{ITD} \) \hspace{1cm} \text{Ibrahim Time Domain}

\( \text{MRITD} \) \hspace{1cm} \text{Multi-Reference Ibrahim Time Domain}

\( \text{RFP} \) \hspace{1cm} \text{Rational Fraction Polynomial}

\( \text{LSCF} \) \hspace{1cm} \text{Least Squares Complex Frequency}

\( \text{UMPA-LOFD} \) \hspace{1cm} \text{UMPA Low Order Frequency Domain}

\( \text{SSI} \) \hspace{1cm} \text{Stochastic Subspace Iteration}

\( \text{CMIF} \) \hspace{1cm} \text{Complex Mode Indicator Function}

\( \text{EMIF} \) \hspace{1cm} \text{Enhanced Mode Indicator Function}

\( \text{FDD} \) \hspace{1cm} \text{Frequency Domain Decomposition}

\( \text{eFDD} \) \hspace{1cm} \text{Enhanced Frequency Domain Decomposition}
Abstract

The Unified Matrix Polynomial Approach (UMPA) was developed to understand the basic similarities among various commercially used experimental modal analysis (EMA) algorithms like LSCE, ERA, PTD, PFD etc. It provided a common framework to develop most of these algorithms that have been developed in isolation, thus making it easier to understand and compare them. This paper extends the concept of UMPA to Operational Modal Analysis (OMA) and illustrates methodologies needed in order to make UMPA suitable for cases where the forces exciting the system are not measured.

1. Introduction

One of the significant contributions of the Unified Matrix Polynomial Approach (UMPA) [1-3] concept to the field of the experimental modal analysis (EMA) was to present the various modal parameter estimation algorithms using a consistent mathematical formulation. This approach not only helped in better understanding of the underlying similarities and differences of the various algorithms, it also provided a common framework to develop these same algorithms which over the years had been developed in isolation.

In recent times, a new technique has emerged which identifies the modal parameters only on the basis of output response data. This technique is referred as Operational Modal Analysis (OMA). The basic difference between the OMA based modal parameter estimation algorithms and the more common EMA parameter estimation algorithms is the fundamental data used. While EMA based algorithms use frequency response functions or impulse response functions (normalized input-output functions in the frequency or time domain), OMA based algorithms use output response power spectrum or correlation functions.

With so many obvious advantages both in terms of developing or understanding the various parameter estimation algorithms and also in understanding the overall parameter estimation process, it is very relevant to extend the concept of UMPA to Operational Modal Analysis. This forms the motivation of the paper where a unified matrix polynomial approach based formulation is reviewed for various OMA algorithms. Section 2 discusses the general modal parameter estimation process and introduces the UMPA model. In section 3, basics of OMA are discussed and UMPA is extended to OMA framework. Section 4 provides the UMPA based mathematical equations of the various time, frequency and spatial domain algorithms and finally a simple case study is provided to show the effectiveness of the UMPA methodology in OMA domain.

2. Modal Parameter Estimation

Matrix equation of motion for a general multi degree of freedom system is given by

\[
[M]\ddot{x}(t) + [C]\dot{x}(t) + [K]x(t) = \{f(t)\}
\]

Where:
- \([M]\) is mass matrix,
- \([C]\) is damping matrix,
- \([K]\) is stiffness matrix,
- \(\{x(t)\}\) is response vector and
- \(\{f(t)\}\) is force vector.

The above equation represents the physical M-C-K model of the system. It is a second order differential equation that can be solved either in time, frequency or Laplace domain. This second order model can be converted into higher order model to handle the case where spatial information is truncated to a size smaller than the number of eigenvalues in the measured data. One way to develop this concept is to obtain the characteristic equation by Laplace transforming Eq. (1). Thus

\[
[M]s^2 + [C]s + [K]X(s) = \{F(s)\}
\]

and the characteristic equation becomes
The partitioned form of above equation can be written as

\[
\begin{bmatrix}
[M_{11}] & [M_{12}] \\
[M_{21}] & [M_{22}]
\end{bmatrix}
- \ldots
\begin{bmatrix}
[M_{1m}] \\
[M_{2m}]
\end{bmatrix}
\begin{bmatrix}
[C_{11}] & [C_{12}] \\
[C_{21}] & [C_{22}]
\end{bmatrix}
- \ldots
\begin{bmatrix}
[C_{1m}] \\
[C_{2m}]
\end{bmatrix}
\begin{bmatrix}
[K_{11}] & [K_{12}] \\
[K_{21}] & [K_{22}]
\end{bmatrix}
- \ldots
\begin{bmatrix}
[K_{1m}] \\
[K_{2m}]
\end{bmatrix}
\begin{bmatrix}
\alpha_2 \\
\alpha_1
\end{bmatrix} = \begin{bmatrix}
\alpha_0
\end{bmatrix} = 0
\]

This equation can be expanded to a higher order matrix polynomial and put in a generic form as

\[
\begin{bmatrix}
\alpha_2 \\
\alpha_1
\end{bmatrix} \begin{bmatrix} 2m \\
2m-1
\end{bmatrix} \begin{bmatrix} s \\
1
\end{bmatrix} + \ldots + \begin{bmatrix}
\alpha_0
\end{bmatrix} = 0
\]

Note that size of \([\alpha]\) is same as the size of the portioned sub matrices and each \([\alpha]\) matrix involves a matrix product and summation of several \([M_{ij}], [C_{ij}]\) and \([K_{ij}]\) sub matrices.

The higher order equation (Eq. 5) has the same eigenvalues as the original second order differential equation (Eq. 1). The general matrix polynomial formulation of the differential equations in the time, frequency and Laplace domain is given by

Time Domain

\[
\begin{bmatrix}
\alpha_m \frac{d^n x(t)}{dt^n} + [\alpha_{m-1}] \frac{d^{m-1} x(t)}{dt^{m-1}} + \ldots + [\alpha_0] x(t) \\
\beta_n \frac{d^n f(t)}{dt^n} + [\beta_{n-1}] \frac{d^{n-1} f(t)}{dt^{n-1}} + \ldots + [\beta_0] f(t)
\end{bmatrix}
\]

Frequency Domain

\[
\begin{bmatrix}
\alpha_m (j\omega)^n + [\alpha_{m-1}] (j\omega)^{m-1} + \ldots + [\alpha_0] X(\omega) \\
\beta_n (j\omega)^n + [\beta_{n-1}] (j\omega)^{n-1} + \ldots + [\beta_0] F(\omega)
\end{bmatrix}
\]

Laplace Domain

\[
\begin{bmatrix}
\alpha_m (s)^n + [\alpha_{m-1}] (s)^{m-1} + \ldots + [\alpha_0] X(s) \\
\beta_n (s)^n + [\beta_{n-1}] (s)^{n-1} + \ldots + [\beta_0] F(s)
\end{bmatrix}
\]

The above described matrix coefficient polynomial forms a good basis to understand the common characteristics of different modal parameter estimation algorithms.

To understand the model further, Eq. (7) is considered. This is the historically used polynomial model for frequency response function \(H(\omega))\). If \(p\) and \(q\) are response and excitation degree of freedoms respectively, Eq. (7) can be written as
\[ H_{pq}(\omega_l) = \frac{X_p(\omega_l)}{F_q(\omega_l)} = \frac{\beta_n(j\omega)^n + \beta_{n-1}(j\omega)^{n-1} + \cdots + \beta_0(j\omega)^0}{\alpha_m(j\omega)^m + \alpha_{m-1}(j\omega)^{m-1} + \cdots + \alpha_0(j\omega)^0} \] (09)

This can be rewritten as

\[ H_{pq}(\omega_l) = \frac{X_p(\omega_l)}{F_q(\omega_l)} = \frac{\sum_{k=0}^{n} \beta_k(j\omega)^k}{\sum_{k=0}^{m} \alpha_k(j\omega)^k} \] (10)

or for a general multiple input, multiple output case

\[ \left[ \sum_{k=0}^{m} (j\omega)^k [\alpha_k] \right] [H(\omega_l)] = \left[ \sum_{k=0}^{n} (j\omega)^k [\beta_k] \right] [I] \] (11)

The size of coefficient matrices is normally \( N_i \times N_i \) or \( N_o \times N_o \) for \([\alpha_k]\) and \( N_i \times N_o \) or \( N_o \times N_i \) for \([\beta_k]\) where \( N_i \) and \( N_o \) are number of input and output degrees of freedom respectively.

This general model corresponds to an AutoRegressive – Moving Average (ARMA(n,m)) model developed from a set of discrete time equations in the time domain. The model, more appropriately, is an AutoRegressive with eXogenous inputs (ARX(n,m)) model. The general matrix polynomial model concept recognizes that both time and frequency domain models generate functionally similar matrix polynomial models. This model which describes both domains is thus termed as Unified Matrix Polynomial Approach (UMPA) [1-3]. Note that Eq. (11) can be repeated at many frequencies \( (\omega_i) \) until the system is sufficiently over determined.

Parallel to above formulation, a time domain model can be developed. For a general multiple input, multiple output case

\[ \sum_{k=0}^{m} [\alpha_k] [x(t_{i+k})] = \sum_{k=0}^{n} [\beta_k] [f(t_{i+k})] \] (12)

For impulse response or free decay data the above equation will reduce to

\[ \sum_{k=0}^{m} [\alpha_k] [h(t_{i+k})] = 0 \] (13)

as forcing can be assumed to be zero for all times greater that zero. Note that \( h(t) \) is impulse response function.

The Unified Matrix Polynomial Approach as explained above provides common framework to most commonly used modal parameter estimation algorithms. This unified perspective provides for easy understanding of the various algorithms such as Complex Exponential Algorithm (CEA) [4,5], Least Squares Complex Exponential (LSCE) [5], Ibrahim Time Domain (ITD) [6,7], Polyreference Time Domain (PTD) [8,9], Polyreference Frequency Domain (PFD) [10-12], Eigensystem Realization Algorithm (ERA) [13-14], Multiple Reference Ibrahim Time Domain (MRITD) [15], Rational Fractional Polynomial (RFP) [16] etc. which over the years have been developed in isolation. Table 1 shows how various commercial modal parameter estimation algorithms fit into UMPA framework. Thus UMPA model helps in understanding the similarities, differences and numerical characteristics of the various modal parameter estimation algorithms by providing a common mathematical structure. [1-3] provide more insights and details of the modal parameter estimation using the unified matrix polynomial approach.

The goal of modal parameter estimation is to obtain the modal model of the system which is defined in terms of complex valued modal frequencies \( (\lambda_l) \), modal vectors \( (\psi_l) \) and modal scaling (modal mass or modal A). However, in case of OMA the modal scaling is not estimated due to lack of input force data. Thus the mode shapes are unscaled mode shapes. In case if modal scaling is desired it is obtained using indirect methods as shown in [17, 18].
Table 1 - UMPA Representations of various EMA algorithms

<table>
<thead>
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<th>High Order</th>
<th>Low Order</th>
<th>Zero Order</th>
</tr>
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<td>Time Domain</td>
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<td>Spatial Domain</td>
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3. Understanding the basics of Operational Modal Analysis

The most fundamental difference between OMA and EMA is the fact that unlike EMA, the input force measurement is not generally available and is not required for OMA. The need for this arises more out of application requirements rather than any significant improvement over the already existing EMA techniques. In situations such as civil structures like bridges, buildings, heavy machinery in operation etc., it is very difficult to excite the structure sufficiently using an artificial excitation system. Even when it’s possible to do so, the cost involved might not be justifiable. Further, testing under simulated lab conditions might yield very different results from the ones obtained under true operational conditions. Situations such as these require an OMA method where dynamic characteristics of the structure are estimated based on the output response data only.

However, for OMA techniques to work properly, there are some key assumptions made that are not obvious at first inspection. The two primary assumptions are as follows: 1) The power spectrum of the input force is assumed to be broadband and smooth. This means that the input power spectra is constant and has no poles or zeroes in the frequency range of interest. 2) The forcing is further assumed to be uniformly distributed spatially. The importance of these assumptions will become clearer in subsequent sections.

If \( \{X(\omega)\} \) is the measured response and \( \{F(\omega)\} \) is the input force, the relationship between them in terms of frequency response function \([H(\omega)]\) is given as follows [19]:

\[
\{X(\omega)\}= [H(\omega)]\{F(\omega)\} \quad \text{(14)}
\]

\[
\{X(\omega)\}^H = \{F(\omega)\}^H [H(\omega)]^H \quad \text{(15)}
\]

Now multiplying Eq. (14) and Eq. (15)

\[
\{X(\omega)\}^H \{X(\omega)\} = [H(\omega)]\{F(\omega)\} \{F(\omega)\}^H [H(\omega)]^H \quad \text{(16)}
\]

or with averaging,

\[
[G_{XX}(\omega)] = [H(\omega)] [G_{FF}(\omega)] [H(\omega)]^H \quad \text{(17)}
\]

where \([G_{XX}(\omega)]\) is the output response power spectra and \([G_{FF}(\omega)]\) is the input force power spectra. Going back to the assumption that the input force spectrum is constant, it is easy to note that the output response power spectra \([G_{XX}(\omega)]\) is proportional to the product \([H(\omega)]H(\omega)]^H\) and the order of output response power spectrum is twice that of frequency response functions. Since \([G_{FF}(\omega)]\) is constant, \([G_{XX}(\omega)]\) can be expressed in terms of frequency response functions as

\[
[G_{XX}(\omega)] \propto [H(\omega)]II[H(\omega)]^H \quad \text{(18)}
\]

or in terms of UMPA model of \([H(\omega)]\) as

\[
G_{XX}(\omega) = \left[ \sum_{k=0}^{n} \left[ \frac{\beta_k}{m} \right] (j\omega)^k \right] \left[ \sum_{k=0}^{m} \left[ \frac{\alpha_k}{n} \right] (j\omega)^k \right]^H \quad \text{(19)}
\]
Further, since \((n < m)\), a partial fraction form of the modal model can be formed for the output power spectrum. This partial fraction model for a particular response location \(p\) and reference location \(q\) is given by

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R_{pqk}^*}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - \lambda_k} + \frac{S_{pqk}^*}{j\omega - \lambda_k^*}
\]  

(20-a)

and can be more conveniently written as [20]

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R_{pqk}^*}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - \lambda_k} + \frac{S_{pqk}^*}{j\omega - \lambda_k^*}
\]  

(20-b)

\[
G_{pq}(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R_{pqk}^*}{j\omega - \lambda_k^*} + \frac{S_{pqk}}{j\omega - \lambda_k} + \frac{S_{pqk}^*}{j\omega - \lambda_k^*}
\]

(20-c)

where \(S_{pqk}\) and \(S_{pqk}^*\) are redefined to incorporate (-1).

Note that \(\lambda_k\) is the pole and \(R_{pqk}\) and \(S_{pqk}\) are the \(k^{th}\) mathematical residues. These residues are different from the residue obtained using a frequency response function based partial fraction model since they do not contain modal scaling factor (as no force is measured). The form of Eq. (20-c) clearly indicates that the roots that will be found from the power spectrum data will be \(\lambda_k, \lambda_k^*, -\lambda_k\) and \(-\lambda_k^*\) for each model order 1 to \(N\).

To formulate a unified matrix polynomial approach for Operational Modal Analysis, Eq. 19 can be rewritten as

\[
\begin{bmatrix}
\sum_{k=0}^{2n} (j\omega)^k \left[\alpha_k \right]
\end{bmatrix}
\begin{bmatrix}
G_{XX}(\omega)
\end{bmatrix} = \begin{bmatrix}
\sum_{k=0}^{2n} (j\omega)^k \left[\beta_k \right]
\end{bmatrix} [I]
\]

(21)

Note that the power spectrum UMPA model is twice the order of the FRF based UMPA model. Further, the coefficient matrices \(\alpha\) and \(\beta\) contains the same system parameters related information twice. This explains Eqs. 20 (a, b, c) which shows that power spectrum data contains the positive and negative poles.

The presence of negative poles can also be explained by means of correlation functions, which are time domain equivalent of power spectrums. Figure 1 shows auto-correlation function of a typical structural response obtained when the structure is randomly excited. The correlation function is a symmetric function. Further, the positive lags give rise to the decaying exponential portion of the correlation function and the negative lags results in the growing exponential portion. There is essentially the same information in both the decaying and growing exponential portions of the correlation function. This again explains the presence of the positive and negative poles as indicated by Eq. 20-c through the use of power spectrums. The positive and negative poles are obtained from the decaying exponential and growing exponential portion of the correlation function respectively.
The high order of the power spectrum based model in comparison to FRF based model causes various disadvantages which makes it more difficult for the frequency domain based algorithms to give good results as they inherently suffer from numerical conditioning problems [21-23]. This problem is not as severe in case of time domain algorithms as one can still work with decaying exponential portion of the correlation function to obtain the modal parameters.

This problem of dealing with the higher order model and presence of positive and negative poles forms the basis of the positive power spectrum which is defined in the frequency domain by the following equation.

\[
G_{pq}^+(\omega) = \sum_{k=1}^{N} \frac{R_{pqk}}{j\omega - \lambda_k} + \frac{R_{pqk}^*}{j\omega - \lambda_k^*}
\]  

(22)

The positive power spectrum is calculated by first inverse Fourier transforming the power spectrum to obtain the correlation functions and then removing the negative lag portion of the correlation function. This is equivalent to multiplying the correlation function with the unit step function in the time domain. The resultant function is then Fourier transformed back to obtain the positive power spectrum. Figure 2 illustrates the process of obtaining the positive power spectrum from the output response data. The advantage of positive power spectrum is that it has the same order as the frequency response functions and also contains all the necessary system information (poles and vectors). Thus UMPA equations can now be applied to positive power spectrum data to perform operational modal analysis of the given system.

The UMPA equivalent equations of Eqs. 12 and 13 for the operational modal analysis can thus be written in terms of positive power spectrum \(G_{XX}^+\) (in frequency domain) and Correlation function \(R_{XX}\) (in time domain) as

\[
\begin{bmatrix}
\sum_{k=0}^{m} (j\omega)^k [\alpha_k] \\
\sum_{k=0}^{n} (j\omega)^k [\beta_k]
\end{bmatrix}
\begin{bmatrix}
G_{XX}^+(\omega) \\
I
\end{bmatrix}
= \begin{bmatrix}
\sum_{k=0}^{m} [\alpha_k] [R_{XX}(t_{i+k})]
\end{bmatrix}
\]  

(23)

(24)

Note that only positive lags of the correlation function are used for the above formulation. In the next section the various algorithms in terms of their UMPA formulation are described.
Figure 2 – Generation of positive power spectrum from output time responses

4. UMPA Formulation of OMA Algorithms

Before discussing the various algorithms, it is important to note that in the case of Operational Modal Analysis only output responses are measured, thus the measured data does not have any typical reference location as is typical of traditional experimental modal analysis where reference locations are the degrees of freedom where input force is provided. In other words there’s no such thing as driving point in case of OMA. However, for the purpose of parameter estimation, certain response locations are chosen as reference locations. These locations are chosen keeping the same considerations as those while choosing the driving points, i.e. reference locations should be the degrees of freedom which excite most modes (or in case of OMA the locations from where most modes can be observed, node points should be avoided, etc.). Unlike the EMA case, though, the reference locations for the OMA case do not mean that an independent excitation has been applied at these degrees of freedom. Therefore, the independent information associated with the reference in the EMA case does not extend to the OMA case. In the discussion that follows, Nref refers to the response locations chosen as reference locations and N0 refers to the output response locations. Also the starting equation in case of time domain algorithms is Eq. 24 and in case of frequency domain algorithms is Eq. 23.

4.1 Time Domain Algorithms

4.1.1 Higher Order UMPA Model

Higher order UMPA algorithms utilize more temporal information in comparison to the spatial information. Thus, the number of response locations is comparatively much higher than the number of reference responses, i.e. N0 >> Nref. The matrix coefficients in this case have the dimension Nref X Nref. Further, if m is model order, the total number of system modes that will be estimated by the model is mNref which is much higher than the required 2N modes of the system. Basic equation for this algorithm is given as

\[
\begin{bmatrix}
[\alpha_1] \\
[\alpha_2] \\
\vdots \\
[\alpha_m]
\end{bmatrix}
\begin{bmatrix}
[R_{xx} (t_{i+1})] \\
[R_{xx} (t_{i+2})] \\
\vdots \\
[R_{xx} (t_{i+m})]
\end{bmatrix}
= -[R_{xx} (t_{i+0})]_{N_{ref} \times N_{o}}
\] (25)
Note that the above equation utilizes zero order coefficient \( \alpha_0 \) normalization. Similar equations can be developed by normalizing other coefficients to come up with different set of solutions. This normalization is very important with respect to where the unwanted poles, associated with the noise in the data, are found. This aspect of the coefficient normalization affects all model solutions (high and low order, time and frequency domain). Every solution will comprise of \( mN_{\text{ref}} \) number of modes out of which \( 2N \) will be genuine system modes and rest will be computational modes. One of the ways to filter out these computational modes is to compare the solution obtained by normalizing various coefficients. The true modes of the system will be retained in each solution but computational modes will differ and can thus be filtered. Once the coefficient matrices are obtained, the roots of the matrix characteristic equation can be found as the eigenvalues of the associated companion matrix \([3]\). As mentioned earlier while working with correlation functions care should be taken to utilize only the positive lag portion of the correlation function.

The popular Polyreference Time Domain (PTD) \([8, 9]\) algorithm is a multi-input, multi-output version of high order UMPA model based algorithm. Similarly, the Complex Exponential and Least Squares Complex Exponential algorithms \([4, 5]\) are SISO and SIMO versions of this model.

### 4.1.2 Lower Order UMPA Model

Lower order algorithms use more spatial information in comparison to temporal information. The matrix coefficients \( \alpha \) have a dimension \( 2N_o \times 2N_o \) (or \( N_o \times N_o \)) and model order \( m \) is 1 (or 2). Thus total number of modes obtained through the algorithm which is more than the required \( 2N \) number of system modes.

Ibrahim Time Domain (ITD) \([6, 7]\), Eigensystem Realization Algorithm (ERA) \([13, 14]\) and Multiple Reference Time Domain (MRITD) \([15]\) algorithms belong to this category of UMPA formulation. In OMA domain, this lower order UMPA formulation is equivalent to the Stochastic Subspace Identification (SSI) algorithm \([24]\) that uses a state space model based on output response correlation functions. The process of obtaining the modes once the coefficient matrices have been found is same as explained in previous section.

Eq. 26 shows the zero order coefficient \( [\alpha_0] \) normalization with \( m = 1 \).

\[
[\alpha_1]_{2N_o \times 2N_o} \begin{bmatrix} \left[ R_{xx} \left( t_{i+1} \right) \right] \\ \left[ R_{xx} \left( t_{i+2} \right) \right] \end{bmatrix}_{2N_o \times N_{\text{ref}}} = - \begin{bmatrix} \left[ R_{xx} \left( t_{i+0} \right) \right] \\ \left[ R_{xx} \left( t_{i+1} \right) \right] \end{bmatrix}_{2N_o \times N_{\text{ref}}}
\]

### 4.2 Frequency Domain Algorithms

#### 4.2.1 Higher Order UMPA Model

The frequency domain equivalent of higher order time domain algorithms can be formulated using the UMPA model in the following manner. This formulation utilizes the positive power spectrum data rather than power spectrum.

\[
[\alpha_1] [\alpha_2] \ldots [\alpha_n] \begin{bmatrix} \beta_1 \quad \beta_2 \quad \ldots \quad \beta_n \end{bmatrix}_{N_{\omega_{max}} \times \left( n+1 \right) N_{\omega_{max}}} = \begin{bmatrix} \left( j\omega_o \right) \beta_1 \left( j\omega_o \right) \beta_2 \ldots \left( j\omega_o \right) \beta_n \end{bmatrix}_{N_{\omega_{max}} \times \left( n+1 \right) N_{\omega_{max}}}
\]

Note that in Eq. 27 the zero order coefficient \( \alpha_0 \) is normalized and this equation can be repeated for other frequencies \( \omega_o \). This model is UMPA equivalent of Rational Fraction Polynomial (RFP) \([16]\) and polyreference least square complex frequency (PLSCF or PolyMAX) \([20, 21, 25]\) algorithms. One of the disadvantages of high order frequency domain algorithms like RFP is that these algorithms involve power polynomials with increasing powers of the frequency. These matrices have Van der Monde form and suffer from poor numerical conditioning problems for wide frequency range and high orders. This
obviously hinders the modal parameter estimation process. Along with limiting the frequency range and reducing the order of the model, normalizing the frequency range and using orthogonal polynomials are some of the methods to reduce this ill-conditioning problem [22]. The polyreference least square complex frequency (PolyMAX) algorithm proposed the use of complex Z mapping and has been shown to have much superior numerical conditioning than other prevalent methods.

4.2.2 Lower Order UMPA Model

Lower order, frequency domain algorithms are basically UMPA based models that generate first or second order matrix coefficient polynomials. Recently the UMPA-LOFD algorithm [23] was proposed for OMA which is a second order \((m=2)\) UMPA model based algorithm. It was shown that the UMPA-LOFD algorithm has good numerical characteristics. The matrix coefficients in this case have \(N_x N_y\) dimensions and thus the total number of modes found is \(2N_x\). Similar to high order frequency domain algorithms this basic equation can be repeated for several frequencies and the matrix polynomial coefficients can be obtained using either \([\alpha_2]\) or \([\alpha_0]\) normalization. The normalized zero order coefficient \([\alpha_0]\) version of this algorithm is shown below

\[
\begin{bmatrix}
[\alpha_1] & [\alpha_2] & [\beta_0] & [\beta_1]
\end{bmatrix}
_{N_x \times 4N_i}
\begin{bmatrix}
(j \omega_i)^2G^+_{xx}(\omega_i) \\
(j \omega_i)^3G^+_{xx}(\omega_i) \\
-(j \omega_i)^5[I] \\
-(j \omega_i)^4[I]
\end{bmatrix}
_{4N_x \times N_i} = -(j \omega_i)^5G^+_{xx}(\omega_i)
_{N_x \times N_i}
\]

(28)

4.3 Spatial Domain Algorithms

Spatial domain algorithms like Complex Mode Indicator Function (CMIF) [26, 27, 30] and its extension Enhanced Mode Indicator Function (EMIF) [28, 29, 30] can be treated as a special case of UMPA model where coefficient matrix has an order zero \((m = 0)\). These algorithms rely only on spatial information and essentially neglect temporal information (spatial information is compared between different temporal solutions). These algorithms utilize the singular value decomposition of the frequency response function matrix at each frequency line to estimate the modal parameters of the system. The Frequency Domain Decomposition (FDD) [31] technique is an extension of CMIF technique in operational modal analysis domain. This technique performs the singular value decomposition on the power spectrum matrix instead of frequency response function matrix. The FDD technique is followed by enhanced Frequency Domain Decomposition (eFDD) [32] technique to estimate the damping and complete the parameter estimation procedure. The FDD-eFDD algorithms were shown to give good results in a number of applications including civil structures, automobiles etc [33-34]. Recently an alternative to eFDD algorithm was proposed which extended the EMIF algorithm to operational modal analysis [35]. This algorithm differs from the eFDD approach in the sense that the parameter estimation is carried out in frequency domain unlike eFDD where the parameter estimation is done in time domain.

It was also shown that for spatial domain algorithms to work satisfactorily, it is necessary that the system is excited completely from a spatial perspective. This means that the source of excitation must be applied at a large number of degrees of freedom (as in wind or wave excitation) rather than at one or two degrees of freedom (as with a shaker or a rotating unbalance source). In all cases the excitation is assumed to be broadband in its frequency content. This is in fact one of the important assumptions made while performing operational modal analysis. A tool called Singular Value Percentage Contribution (SVPC) plot was also proposed which helps in determining whether the system is being excited locally or spatially uniformly. This tool makes it possible to use the CMIF plot even in cases where the system is not spatially well excited.

5. Case Study: Lightly Damped Circular Plate

Having developed various OMA algorithms using the UMPA formulation, these algorithms are now applied to a simple lightly damped circular plate. A circular plate due to its peculiar geometry is a good experimental structure to test these algorithms as a lot of closely spaced modes are present. The plate is excited randomly all over its
surface by means of an impact hammer. A total of 30 accelerometers are placed over the plate to measure the output response (Figure 3).

![Figure 3: Experimental set up for the lightly damped circular plate](image)

The modal parameters obtained using various OMA algorithms are shown in the table 2. The modal parameters obtained using the various UMPA formulated OMA algorithms show very good agreement. The purpose of the study however is not to comment on the performance of the individual algorithms but the fact these algorithms can be developed very easily if the underlying unified concept is understood. UMPA methodology aid greatly in this regard and this underlines its utility and effectiveness.

**Table 2: Modal Parameters estimated using various UMPA formulated OMA algorithms**

<table>
<thead>
<tr>
<th>System modes using EMA</th>
<th>UMPA Higher Order Time Domain (PTD)</th>
<th>UMPA Lower Order Time Domain (ERA)</th>
<th>UMPA Higher Order Frequency Domain (RFP or PLSCE)</th>
<th>UMPA Lower Order Frequency Domain (PFD or LOFD)</th>
<th>UMPA Higher Order Frequency Domain with complex z mapping (RFP-z or PolyMAX)</th>
<th>UMPA Lower Order Frequency Domain with complex z mapping (PFD-z or LOFD 2)</th>
<th>UMPA Zero Order Spatial Domain (CMF-EMIF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damp</td>
<td>Freq</td>
<td>Damp</td>
<td>Freq</td>
<td>Damp</td>
<td>Freq</td>
<td>Damp</td>
<td>Freq</td>
</tr>
<tr>
<td>0.268</td>
<td>66.691</td>
<td>0.611</td>
<td>66.439</td>
<td>0.611</td>
<td>66.436</td>
<td>0.663</td>
<td>56.462</td>
</tr>
<tr>
<td>0.285</td>
<td>57.194</td>
<td>0.623</td>
<td>57.191</td>
<td>0.619</td>
<td>57.197</td>
<td>0.669</td>
<td>57.214</td>
</tr>
<tr>
<td>0.312</td>
<td>96.377</td>
<td>0.838</td>
<td>96.377</td>
<td>0.831</td>
<td>96.361</td>
<td>0.838</td>
<td>96.363</td>
</tr>
<tr>
<td>0.412</td>
<td>132.101</td>
<td>0.351</td>
<td>131.702</td>
<td>0.338</td>
<td>131.705</td>
<td>0.353</td>
<td>131.664</td>
</tr>
<tr>
<td>0.147</td>
<td>132.665</td>
<td>0.304</td>
<td>132.609</td>
<td>0.311</td>
<td>132.601</td>
<td>0.312</td>
<td>132.743</td>
</tr>
<tr>
<td>0.243</td>
<td>219.932</td>
<td>0.302</td>
<td>219.094</td>
<td>0.301</td>
<td>219.092</td>
<td>0.299</td>
<td>219.37</td>
</tr>
<tr>
<td>0.216</td>
<td>220.352</td>
<td>0.371</td>
<td>221.075</td>
<td>0.37</td>
<td>221.08</td>
<td>0.367</td>
<td>221.35</td>
</tr>
<tr>
<td>0.214</td>
<td>231.172</td>
<td>0.252</td>
<td>230.545</td>
<td>0.28</td>
<td>230.865</td>
<td>0.257</td>
<td>230.665</td>
</tr>
<tr>
<td>0.157</td>
<td>232.077</td>
<td>0.212</td>
<td>232.102</td>
<td>0.22</td>
<td>232.086</td>
<td>0.227</td>
<td>232.391</td>
</tr>
<tr>
<td>0.089</td>
<td>352.397</td>
<td>0.152</td>
<td>351.214</td>
<td>0.144</td>
<td>351.18</td>
<td>0.151</td>
<td>351.69</td>
</tr>
<tr>
<td>0.174</td>
<td>355.506</td>
<td>0.224</td>
<td>355.303</td>
<td>0.222</td>
<td>355.283</td>
<td>0.22</td>
<td>355.79</td>
</tr>
<tr>
<td>0.118</td>
<td>374.554</td>
<td>0.273</td>
<td>373.424</td>
<td>0.27</td>
<td>373.382</td>
<td>0.27</td>
<td>373.941</td>
</tr>
<tr>
<td>0.176</td>
<td>377.569</td>
<td>0.239</td>
<td>376.99</td>
<td>0.242</td>
<td>377.013</td>
<td>0.236</td>
<td>377.508</td>
</tr>
<tr>
<td>0.313</td>
<td>414.414</td>
<td>0.245</td>
<td>411.168</td>
<td>0.239</td>
<td>411.138</td>
<td>0.241</td>
<td>411.729</td>
</tr>
<tr>
<td>0.209</td>
<td>498.891</td>
<td>0.22</td>
<td>484.72</td>
<td>0.22</td>
<td>484.627</td>
<td>0.221</td>
<td>485.408</td>
</tr>
</tbody>
</table>
6. Conclusions

The concept of Unified Matrix Polynomial Approach (UMPA) is extended to Operational Modal Analysis. It is shown how various time, frequency and spatial domain OMA algorithms can be formulated using the UMPA model. Emphasis is placed on understanding the basic difference between traditional Experimental Modal Analysis and output-only Operational Modal Analysis, the various assumption made in case of OMA and how the fundamental data (correlation functions and power spectrums) should be used in order to utilize the UMPA model for the purpose of parameter estimation in case of OMA. It is revealed that understanding the underlying basic polynomial model not only helps in theoretical development of various algorithms but also provides a common framework which makes it much easier and simpler to understand these algorithms. It is important to reiterate that assumptions concerning the nature of the assumed excitation (smooth and broadband in frequency, spatially well distributed, etc.) are critical to the success of OMA methods.

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References


