Identifying Nonlinear Parameters for Reduced Order Models. Part I: An Analytical Comparison

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Abstract

Assembling nonlinear dynamic models of structures is the goal of numerous research and development organizations. Such a predictive capability is required for the development of advanced, high-performance aircraft structures. Specifically, the ability to predict the response of complex structures to aero-acoustic loading has long been a United States Air Force (USAF) goal. Sonic fatigue has plagued the Air Force since the advent and adoption of the aircraft turbine engine. While the sonic fatigue problem has historically been a maintenance one, predicting nonlinear dynamic response is crucial for future aerospace vehicles. Decades have been spent investigating the dynamic response and untimely failures of aircraft structures, yet little work has been accomplished towards developing practical nonlinear prediction tools. The aim of this paper and a companion one is to present a novel means of assembling nonlinear reduced order models using experimental data and an analytical basis. This paper (Part I) outlines a unique extension of a recently introduced nonlinear identification method; Nonlinear Identification through Feedback of the Outputs (NIFO). The novel extension, or the modal NIFO method, allows for a ready means of identifying nonlinear parameters in reduced order space using measured data. The nonlinear parameters are then used in the assembly of reduced order models, thus providing researchers with a means of conducting predictive studies prior to expensive and questionable experimental efforts. In this paper, an analytical experiment is conducted, using the results from a dynamic nonlinear Finite Element (FE) model, demonstrating the practicality of the method in generating useful nonlinear reduced order models. Nonlinear stiffness parameters were successfully identified for a multiple degree-of-freedom (MDOF) nonlinear reduced order model. The nonlinear coefficients compare well with previously published analytical studies of the beam.

1. Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>f(t)</td>
<td>Time Domain Forcing Function</td>
</tr>
<tr>
<td>ρ_r</td>
<td>Modal Coordinate in Physical Space</td>
</tr>
<tr>
<td>ξ_r</td>
<td>Modal Damping Value</td>
</tr>
<tr>
<td>ω_r</td>
<td>Modal Natural Frequency</td>
</tr>
<tr>
<td>φ_r</td>
<td>Basis Vector</td>
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<tr>
<td>θ_r</td>
<td>Modal Nonlinear Expansion Terms</td>
</tr>
<tr>
<td>t, ω</td>
<td>Time, Frequency</td>
</tr>
<tr>
<td>x, ẋ, ¨x</td>
<td>Displacement, Velocity, Acceleration in Physical Space</td>
</tr>
<tr>
<td>Θ(ω)</td>
<td>Nonlinear Modal Coupling Terms in the Frequency Domain</td>
</tr>
<tr>
<td>A_r</td>
<td>Nonlinear Coefficient</td>
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<td>A_r̃</td>
<td>Scaled Nonlinear Coefficient</td>
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<tr>
<td>B_r(ω)</td>
<td>Modal Impedance Vector</td>
</tr>
<tr>
<td>F_r</td>
<td>Fourier Transform</td>
</tr>
<tr>
<td>F_r(ω)</td>
<td>Frequency Domain Forcing Function</td>
</tr>
<tr>
<td>H(ω)</td>
<td>Linear Modal Frequency Response Function</td>
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2. Introduction

The USAF is interested in developing a nonlinear dynamic response prediction capability. The capability is crucial for developing structures intended to withstand the intense acoustic environments experienced by structures exposed to engine induced and aero-acoustic loading. Recently there have been numerous attempts, primarily analytical, to provide research tools capable of predicting the response of aircraft structures to dynamic loading. One of the key assumptions of any analytical based structural prediction method is the ability to accurately model the structure, including boundary conditions and material properties. Given the complexity of future aerospace structures, confidence based solely in modeling is suspect at best. The USAF is developing a method of assembling nonlinear reduced order models directly from experimental measurements. Part I of this paper demonstrates this modal NIFO method using an analytic ‘experiment’, while Part II demonstrates its practicality method using a well characterized experiment. The modal NIFO method allows for rapid modification of the assumed nonlinear form thus providing researches with a means of identifying a full spectrum of reduced order nonlinear models. This is a significant difference from many of the methods in the literature, where the identification of simultaneous nonlinearities or the quick adoption of various forms is prohibitive. Another key feature of the modal NIFO method presented herein is the ability to identify the nonlinear parameters directly with no iteration or presumptions as to appropriate initial values. Also, the method is free from restrictive loading conditions, i.e., harmonic balance or the method of multiple scales. In fact the results presented in this study and the companion one are based upon broadband random inertial loading. Three of the underlying assumptions in the method are first, the nonlinearities are indeed exercised, second an underlying or nominal linear system exists, and third, the system nonlinearities can be adequately described in reduced-order space.

A review of the literature reveals that the development of reduced order identification methods is a growing technology field. This paper will begin with a discussion of the purely analytical approaches. The idea of generating reduced order models based solely on FE results has been studied extensively by Nash [1], Shi and Mei [2], McEwan et. al. [3,4,5], Hollkamp et. al. [6,7], Migneot et. al. [8], Przekop and Rizzi [9], and Muravyov and Rizzi [10]. In all of these cases, nonlinear reduced order models are assembled using analytically identified or derived nonlinear coefficients, thus allowing for computationally tractable prediction models. This idea of reduced order models has seen recent interest in order to address a problem common to aircraft designers. To model the response of a realistic aircraft structure to realistic loading requires thousands if not millions of FE degrees-of-freedom. The successful nonlinear dynamic analysis of such a model is questionable, even with high-performance computers. What is required is a computationally efficient and tractable means of analyzing complex structures. The intent of Nash [1] and Shi and Mei [2] was to arrive at nonlinear reduced order models directly through the manipulation of the FE stiffness matrices. In particular, Shi and Mei [2] use only structural bending modes in their basis to describe the response of beam and plate structures. Their in-plane DOFs are assumed not to respond dynamically and are thus described in terms of the transverse DOFs. The resulting geometric nonlinear coefficients are therefore directly evaluated. Two issues preclude widespread use of this approach; first, the condensation approach is limited to structures whereby in-plane and transverse DOFs are separable, and second it is doubtful that commercial FE algorithms could be easily utilized, as the method requires direct modification and manipulation of the stiffness matrices.

The basic idea common to the remaining or ‘indirect’ methods has focused on using generic FE algorithms to identify nonlinear stiffness coefficients indirectly. In order to capitalize on the convenience and flexibility of commercial FE algorithms, McEwan [3,4,5] devised a means of identifying geometric nonlinear coefficients from a series of nonlinear FE static analyses. Using those identified coefficients and the results of FE eigensolutions, a reduced order model can be assembled. Hollkamp et. al. [6] compared the results of a well characterized clamped-clamped beam experiment to the results of reduced order models generated using various FE based methods. Hollkamp [7] also presented an in-depth discussion of FE derived nonlinear reduced order modeling methods. In both references, locally developed algorithms were compared to those developed by McEwan [3,4,5], Migneot et. al. [8], Przekop and Rizzi [9] and Muravyov and Rizzi [10]. The nonlinear coefficients of the various methods, for both single and two DOF models were presented. Some of the principle issues discussed include the appropriate selection or more accurately the appropriate assumptions when assembling a basis set for this type of problem. All of the indirect methods approach the problem from a slightly different perspective, but attempt to capture the sonic fatigue type response of thin skinned aircraft structures, or the modeling of the in-
plane motion through an appropriate basis. It was determined that the method of McEwan [3,4,5] was the most appropriate for this class of problems, as there is no distinction between the in-plane and transverse basis vectors. The in-plane motion of beams or panels is captured through an implicit condensation similar to the method previously discussed. This is an important point, as a similar approach is presented here, i.e., implicit condensation of the in-plane or membrane basis vectors is conducted via experimental identification. A limitation with these reduced order analytical methods is in the tractable form of the assumed nonlinearities. In other words, they are necessarily limited to a stiffness dependent or static form in lieu of computationally expensive nonlinear full-model dynamic analyses, the very analysis the researchers are attempting to avoid. For example, in the work of Hollkamp [6,7], a full suite of static nonlinear FE analyses is conducted to provide the data necessary for the respective identification methodology. The intent of the analytical based methods is to assemble nonlinear reduced order models without having to rely on computationally expensive and oftentimes restrictive dynamic analyses. While the benefits of utilizing commercial FE algorithms to identify nonlinear parameters are obvious, the modal NIFO method presented in this study is not limited solely to static nonlinearities, but again, is adaptable to a variety of nonlinear forms. Thus, the method presented in this study is assumed to be complimentary to the analytical methods described previously.

The analytical work in Part I, and the experimental work detailed in Part II is based on the previously published experimental study of Gordon et. al. [11]. This experimental work considered the results of a well characterized clamped-clamped beam as a means of exercising developing analytical nonlinear reduced order methods. Results of the published work include dynamic displacement and strain results for broadband random inputs. Additionally, experimental estimates of a cubic Duffing parameter for a SDOF model were also presented. To date, there has been no published study of a truly practical and broadly applicable experimental reduced order nonlinear identification method, particularly one compared with a well characterized experiment. There has been a significant amount of work recently in this area, but as will be discussed in Part II, the respective efforts require iterative schemes, hybrid modeling procedures, specific loading conditions, significant computational expense, multiple identification scenarios and generally complicated procedures. This effort presents a straightforward means of generating nonlinear reduced order models directly from raw experimental data utilizing the convenience of an analytically derived basis, although the actual source of the basis is incidental. The study is particularly important for the design and analysis of aircraft structures experiencing severe random acoustic loading. In Part I, analytical studies were conducted on a well characterized clamped-clamped beam setup. The experiment has been demonstrated to exhibit the form of nonlinear, amplitude dependent response typical of those structures experiencing acoustic fatigue. Once the identification was accomplished, the nonlinear parameters were then used in the assembly of a nonlinear MDOF reduced order model. The results of the model compared favorably with analytical results at a more severe loading scenario.

3. Background

This modal NIFO method was developed to determine a simple analytical model useful for prediction that adequately describes the behavior of an aircraft-like structure. This method does not require an iterative scheme, nor are the individual mass, stiffness and damping parameters required or directly identified. Further, the method is a spectral one where nonlinear modal coefficients and the nominal linear system FRFs are identified in a least squares sense, thus the computational time required for the identification is negligible. The method is flexible, in that the proposed form of the nonlinearities can be easily modified and adapted to the problem at hand. Additionally, multiple nonlinearities of significantly different order can be identified simultaneously. The development of this method is based upon the assumption that the nonlinear vibratory response of a structure can be adequately described by combinations of normal modes. As will be demonstrated, the nonlinear reduced order equations remain coupled; functions of the reduced order or modal coefficients.

The NIFO method of Adams and Allemang [12,13] is an elegant means of identifying nonlinear structural parameters as well as the underlying, or nominally linear parameters in a single analysis step. One significant contribution of the method is the efficient use of spatial and temporal information in the characterization of nonlinear parameters. This use of spatial information allows for the identification of nonlinear parameters at both forced as well as unforced DOFs. The method introduced here is based upon the NIFO method, with a significant difference; the nonlinear dynamic equations used in the derivation are presented in a reduced order sense, and further, result in a mathematical representation of the structure of interest useful for prediction purposes. Consider Eq. (1), a representation of a general system equation of motion written in reduced order form:
\[ \ddot{\mathbf{p}}_r(t) + 2\xi_r \omega_r \dot{\mathbf{p}}_r(t) + \omega_r^2 \mathbf{p}_r(t) + \mathbf{\theta}_r(p_1(t), p_2(t), \ldots, p_n(t)) = \phi^T f(t) \]  

(1)

where \( \mathbf{p}_r \) represents the generalized or modal coordinate, \( \zeta_r \) the modal damping, \( \omega_r \) the natural frequencies, \( f(t) \) the applied force in physical coordinates, \( \phi \) the modal or basis vector, and \( \mathbf{\theta}_r \) the nonlinear modal components. Unlike a linear system, the equations represented in generalized coordinates are now coupled through the \( \mathbf{\theta}_r \) terms. It is important to point out that the appropriate nonlinear form of \( \mathbf{\theta}_r \) depends on the application in question.

In the present paper, the assumed form of the nonlinearities will be cubic, based upon previous studies of aircraft structural response [6,7]. Thus, the nonlinear modal contribution takes the following form:

\[ \mathbf{\theta}_r = \sum_{i=1}^{n} A_r(i,i,i) p_i^3(t) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left\{ A_r(i,i,j) p_i^2(t) p_j(t) + A_r(i,j,j) p_i(t) p_j^2(t) \right\} \]  

(2)

where the \( A_r \) terms denote the nonlinear coefficients of interest. Applying the Fourier Transform to Eqs. (1) and (2) results in the following modal impedance equation:

\[ B_r(o) \mathbf{p}_r(o) + \Theta_r(o) = F_r(o) \]  

(3)

where \( B_r(o) \) is the linear modal impedance matrix, \( \mathbf{p}_r(o) \) is the \( r \)-th modal coordinate in the frequency domain, \( F_r(o) \) is the modal force in the frequency domain, and \( \Theta_r(o) \) represents the nonlinear modal coupling terms transformed into the frequency domain. Note that by using the system normal modes in the transformation from physical to modal space, the linear impedance matrix is uncoupled, while the \( \Theta_r(o) \) terms remain coupled, as displayed in Eq. (2). Consider the following example where \( r = 1 \) in a 2-mode expansion of Eq. (3), with the nonlinear terms moved to the RHS:

\[ B_1(o) \mathbf{p}_1(o) = F_1(o) - A_1(1,1,1) \tilde{F}[p_1^3(t)] - A_1(2,2,2) \tilde{F}[p_2^3(t)] - \ldots \]  

\[ \ldots - A_1(1,1,2) \tilde{F}[p_1^2(t) p_2(t)] - A_1(1,2,2) \tilde{F}[p_1(t) p_2^2(t)] \]  

(4)

where \( \tilde{F}[] \) denotes the Fourier Transform. It is also convenient to rewrite Eq. (1) in physical space by introducing the respective physical DOF for the modal expansion term. For example, in this study the structural point of interest will be the beam midpoint or center. Therefore, the beam center transverse displacement can be represented as a function of the retained modes of interest:

\[ x_{\text{center}} = \sum_{i=1}^{n} \phi_{\text{center}}^i p_i(t) \]  

(5)

Further, for the 2-mode expansion example, considering only the displacement at the beam center, Eq. (5) can be represented as:

\[ x_{\text{center}} = \phi_{\text{center}}^1 p_1(t) + \phi_{\text{center}}^2 p_2(t) = x_1 + x_2 \]  

(6)

Therefore, utilizing Eqs. (2) and (6), Eq. (1) with \( i = 1 \) can now be represented as follows:

\[ \ddot{x}_1 + 2\xi_1 \omega_1 \dot{x}_1 + \omega_1^2 x_1 + A_1(1,1,1) \phi_{\text{center}}^1 x_1^3 + A_1(1,1,2) \phi_{\text{center}}^1 \phi_{\text{center}}^2 x_1^2 x_2 + A_1(1,2,2) \phi_{\text{center}}^2 x_1^2 x_2^2 + \ldots \]  

\[ \ldots + A_1(2,2,2) \phi_{\text{center}}^2 x_2^3 = \phi_{\text{center}}^1 \phi_{\text{center}}^2 f(t) \]  

(7)

Note that in Eq. (7), the nonlinear coefficients are scaled by the respective basis set components. Continuing with the 2-mode expansion example and pre-multiplying both sides of Eq. (4) by the linear modal FRF matrix, the following set of equations result:
\| \mathbf{P}_1(\omega) \| = \begin{bmatrix} H_1(\omega) & H_1(\omega)A_1(1,1) & H_1(\omega)A_1(2,2,2) \\ & \vdots & \vdots \\ H_1(\omega)A_1(1,1,2) & H_1(\omega)A_1(1,2,2) \end{bmatrix} \begin{bmatrix} F(\omega) \\ \mathbf{F}p_1(t) \\ \mathbf{F}p_2(t) \end{bmatrix} = \begin{bmatrix} F(\omega) \\ \mathbf{F}p_1(t) \\ \mathbf{F}p_2(t) \end{bmatrix} \begin{bmatrix} \mathbf{P}_1(\omega) \end{bmatrix}

\text{where the unknown nonlinear coefficients are post-multiplied by the measured input and responses. A similar equation in physical space results by utilizing Eq. (7) in the transformation to the frequency domain. In this instance, the } \mathbf{P}_1(\omega) \text{ terms are replaced by the respective physical ones.}

\begin{align*}
\{\mathbf{X}_1(\omega)\} &= \begin{bmatrix} H_1(\omega) & H_1(\omega)A_1(1,1,1) & H_1(\omega)A_1(2,2,2) \\ & \vdots & \vdots \\ H_1(\omega)A_1(1,1,2) & H_1(\omega)A_1(1,2,2) \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}
\end{align*}

\text{where in Eq. (9), the unknown nonlinear coefficients are now scaled by the appropriate basis set components. Note, } \{\mathbf{X}_1(\omega)\} \text{ in Eq. (9) is not the physical response of the DOF 1, but the physical contribution of mode 1 to the total response as defined in Eq. (6). Since experimental data is recorded in the physical domain, it will be necessary to transform or filter the data. This is accomplished via the following equation:}

\begin{align*}
\{\mathbf{p}(t)\} &= \begin{bmatrix} \mathbf{\phi}^T \mathbf{\phi} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\phi}^T \{\mathbf{x}(t)\} \end{bmatrix} = \begin{bmatrix} \mathbf{\phi}^T \{\mathbf{x}(t)\} \end{bmatrix}
\end{align*}

\text{where } \begin{bmatrix} \mathbf{\phi}^T \mathbf{\phi} \end{bmatrix}^{-1} \text{ represents the Moore-Penrose pseudo-inverse of the proposed basis set, providing a best-fit solution in a least squares sense. It was discovered that the transformation from the modal coordinate of Eq. (10) to the pseudo-physical coordinate of Eqs. (6) and (7) circumvented numerical problems during the least-squares implementation of Eq. (9).}

\text{Note that the measured nonlinear response is a function of the underlying linear system as well as the unmeasured nonlinear feedback forces, the basis of the NIFO method. As previously discussed, the linear modal FRFs and nonlinear parameters can now be estimated in a single step via a least-squares formulation of Eqs. (8) or (9). The benefits of the modal NIFO method are now obvious; 1) estimates of the underlying linear modal FRF and nonlinear coefficients are achieved in a single step versus the multi-step approaches of many of the previously described methods, 2) the method results in a series of SDOF linear modal FRFs, themselves appropriate for identification in modal space, 3) because NIFO is a frequency domain method, the linear and nonlinear portions of the equations are essentially uncoupled, and 4) the method is easily adaptable, thus providing researchers with a rapid means of attempting various nonlinear trial functions when various nonlinear elements are suspected or the appropriate form is unknown. The difficulty in obtaining accurate estimates of the nonlinear coefficients lies with the } a \text{ priori knowledge of the form of the nonlinearity, } \Theta(\omega). \text{ Further, as the problem is a nonlinear one, it is understood that unique estimates can not be obtained. What is sought is the best nonlinear modal model useful for prediction purposes. Without explicit understanding of the form of the nonlinearities, general trial functions can be proposed, but numerical issues can arise without care.}

4. Analytical Investigation
The analytical experiment was conducted on a FE model of the aforementioned clamped-clamped beam. The model considered in this investigation is displayed in Fig. 1.

As previously mentioned, this beam model was developed specifically to exercise developing analytic prediction methods. This particular beam configuration was selected in the earlier referenced studies because it exhibits the form of nonlinear response typical of thin aircraft panel type structures. A 20-element symmetric model of the beam was created using nonlinear, large displacement beam elements. Further, linear springs, the values of which were obtained through experimental characterization, were utilized to model the boundary conditions. The loading considered was broadband Gaussian random base excitation. By utilizing a flat input spectrum between 20 and 500 Hz, the first two symmetric bending modes of the beam were excited, 77.7 and 420 Hz respectively. The nonlinear dynamic response of the FE model was used as a 'truth model', as well as input to the modal NIFO method to recreate the beam model in reduced order space. A Newmark-Beta numerical integration technique, with a time step of 1.0e-4 seconds was used to generate the nonlinear dynamic response. 120 seconds of displacement response was obtained for each of five loading scenarios: 0.5, 1.0, 2.0, 4.0 and 8.0g's RMS. Proportional damping was introduced with values of 0.3% and 0.5% for the first two symmetric bending modes respectively. These proportional damping values were obtained from experimental studies of the beam. The Power Spectral Density (PSD) of the beam midpoint response to increasing random input is displayed in Fig. 2. Note the obvious nonlinearities evidenced by the broadening and shifting of the peaks.
Figure 2: Response of beam midpoint to random base-motion inertial loading. (thin solid line: response to 0.5g; dashed line: response to 2g; bold solid line: response to 8g).

4.1 Analytical MDOF Nonlinear Parameter Estimation

A 2-DOF modal model was identified using results from the 8g loading scenario as input to Eq. (9). After reviewing estimates derived from the various response scenarios, the 8g case was deemed the most appropriate, based upon the nominal linear FRF estimate and resulting comparison with the analytical FRF. As explained in the lead-up to Eq. (4), a 2-DOF modal model, cubic in stiffness, will result in eight nonlinear coefficients. The identification is accomplished on a mode-by-mode basis, with four parameters identified for each mode, the $A_i$'s in Eqs. (2) and (4). Similar frequency domain conditioning parameters were used for the MDOF case, namely 291 averages, a Hanning Window, 50% overlap and a blocksize of 4096. Graphical results of the estimation process are displayed in Figs. 4 and 5.
Figure 3: MDOF Mode 1 estimates: (a) nominal linear FRF (solid line: low-level ‘linear’ FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: $A_1^{(1,1,1)}$; bold dashed line: $A_1^{(1,1,2)}$; solid line: $A_1^{(1,2,2)}$; dashed line $A_1^{(2,2,2)}$).

Figure 4: MDOF Mode 2 estimates: (a) nominal linear FRF (solid line: low-level ‘linear’ FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: $A_2^{(1,1,1)}$; bold dashed line: $A_2^{(1,1,2)}$; solid line: $A_2^{(1,2,2)}$; dashed line $A_2^{(2,2,2)}$).
Spectral averages of the estimated nonlinear coefficients along with values obtained using the \textit{Direct Evaluation} [6] FE based identification method are presented in Table 1. The coefficients displayed in Table 1 represent the nonlinear coefficients in physical coordinates, as displayed in Eq. (9). In other words, they are scaled by the particular modal basis component described in Eq. (7). The estimates of both modes compare quite favorably with the results derived using the published method [6]. Consider the nonlinear coefficient estimates in Fig. 5. There is negligible deviation from the average values displayed in Table 1, indicating a high confidence in the estimates. In contrast, consider the region near the first mode peak in Fig. 4, as well as the nonlinear coefficient estimates at nearly twice the fundamental frequency. Recall that for this study, only a cubic nonlinear form was assumed for $\theta_r$. It is supposed that this region is a byproduct of the numerical integration. For this analytical study, all nodal points and associated FE beam degrees-of-freedom were used in the transformation, a situation not possible with experimental analyses. It is clear from these figures, that the response is dominated by the first mode. Thus, estimates for mode 2 were not expected to be as accurate nor as influential as those estimated for the first mode. The effect of the errors in the second mode dominated coefficients was insignificant, as will be demonstrated in the following discussion on prediction.

4.3 Analytical Response Prediction

Given the estimates of the nonlinear coefficients, a modal model was assembled and used for prediction purposes. In this study, only the MDOF results were used for prediction. Again, a Newmark-Beta numerical integration scheme with a time-step of $1.0 \times 10^{-4}$ seconds was used. It is important to point out that calculating 120 seconds of data using the aforementioned modal models required approximately 300 seconds versus the 2-4 hours for comparable full-model, direct integration of this particular and relatively simple nonlinear FE beam model. Thus, one of the benefits of obtaining reduced order models is obvious – significant computational savings. Results of the prediction for the 4 and 8g input cases are displayed in Fig. 6.

![Figure 5: Analytical prediction using identified nonlinear coefficients. (solid line: 4g analytic load case; dashed line: 4g predicted load case; bold solid line: 8g analytic load case; bold dashed line: 8g predicted load case).](image)

Root mean square (RMS) values of the response for all of the loading cases are presented in Table 2. Again, broadband random input excitation was used with the MDOF estimated modal model. Both the displacement PSD and RMS values clearly indicate excellent agreement with the analytic results.
5. Conclusion and Future Work

A novel approach was presented for the estimation and generation of nonlinear modal models. Accurate estimates of nonlinear reduced order coefficients were obtained for a MDOF analytical modal model. Excellent agreement was noted between the analytical identification and previously published results. Further, the MDOF modal model was used for prediction purposes, again with favorable comparisons to the nonlinear FE results. Future analytical studies will consider models incorporating additional nonlinear forms. Further, the uncertainty associated with the nonlinear coefficient estimates, as a function of frequency, and the effect on response prediction will also be investigated.

Table 1: 2-DOF Nonlinear Estimates

<table>
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<tr>
<th>Direct Evaluation Coefficients [6] (in (^2) s(^{-2}))</th>
<th>Estimated Coefficients (in (^2) s(^{-2}))</th>
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<tr>
<td>(A_1^{(1,1,1)} = 1.84E+8)</td>
<td>(A_1^{(1,1,1)} = 1.76E+8)</td>
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<td>(A_1^{(1,1,2)} = 4.82E+8)</td>
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Table 2: RMS Displacement Values.

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<th>Load (g)</th>
<th>Experimental Results (in)</th>
<th>Prediction using estimated coefficients (in)</th>
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<td>0.5</td>
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<td>8.0</td>
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