I hereby recommend that the thesis prepared under my supervision by Jan M. Leuridan entitled SOME DIRECT PARAMETER MODEL IDENTIFICATION METHODS APPLICABLE FOR MULTIPLE INPUT MODAL ANALYSIS be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

Approved by:

[Signatures]
SOME DIRECT PARAMETER MODEL IDENTIFICATION METHODS
APPLICABLE FOR
MULTIPLE INPUT MODELL ANALYSIS

A dissertation submitted to the

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ABSTRACT

In search for better procedures which identify an experimental modal model from measurement data, some new analysis methods, designated direct parameter model identification methods, are developed. These methods are designed to identify an experimental modal model by modelling the measurement data directly using the constitutive differential equations. The parameters in a frequency domain representation, or in a discrete time finite difference approximation, of the differential equations are estimated, using multiple input multiple output data simultaneously. A modal model is then obtained by uncoupling the estimated form of the differential equations. This enables the identification of experimental modal models with highly coupled and pseudo-repeated modes.

In applying the developed methods, frequency response functions, or impulse response functions, free decay data or forced response data can be used. The procedures developed for analyzing time domain data are shown to be numerically better conditioned and to execute faster than the corresponding procedures for frequency domain data. In the time
domain, sampled force input and response sequences can also be analyzed directly, therefore avoiding any truncation errors introduced by a finite discrete transform between time and frequency domain.

Numerous simulated test cases are discussed to demonstrate the various characteristics of the developed direct parameter model identification methods, in particular the capability to obtain improved experimental modal models by simultaneous analysis of multiple input data and direct time domain processing. The applicability on experimental data is demonstrated using impulse response functions for several reference locations of a circular plate structure and an aircraft structure, both structures exhibiting pseudo-repeated modes.
PREFACE

This dissertation deals with some new vibration data analysis methods that have been developed in search for improved procedures to derive an experimental modal model from test data. The developed methods, designated as direct parameter model identification methods, identify a modal model by modelling the response directly using the characteristic differential equations. In a sense these methods address the problem of identifying effective dynamic matrices that, after coordinate transformation, yield the modal model. The developed procedures allow for the simultaneous analysis of multiple input response data, a characteristic that is found to be essential for identifying an improved modal model from test data.

The dissertation is organized in four chapters. The first chapter provides a general background on parameter identification methods for experimental modal analysis, and a discussion of motives for the present research effort. The second chapter proceeds with the theoretical development of a general direct parameter model and applicable estimation procedures, both in the frequency and time domain. The
latter procedures are found to be a lot more attractive, both for their numerical stability and for the applicability of fast recursive solution methods. Applications of the developed methods on analytical and experimental data are discussed in the third chapter. It is demonstrated that the simultaneous analysis of multiple input response data yield an improved experimental modal model, especially when highly coupled and pseudo-repeated modes are present. Finally, the fourth chapter summarizes conclusions and recommendations for further research.

To simplify referencing of the numerous equations throughout the dissertation, the equations have been numbered absolutely within every main section of a chapter. To reference equations between different sections, the equation number is preceded by the section number. For example, Equation (100) of Section (2.3) is in other sections referenced as Equation (2.3.100).

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SYMBOLS

Matrix Notation

[...] brackets enclose matrix expressions

{..} braces enclose vector expressions

A upper case letters indicate matrices

(n,m) dimension indication of a matrix with n rows and m columns

(A)_{i,j} element on crossing of row i and column j in a matrix A. When no ambiguity exists, the notation using the corresponding lower case letter is used and brackets are omitted, a_{i,j}

\{A\}_{i} column i of a matrix A

[A]_{i} row i of a matrix A

[A]\_{i} row or column partition of A (dependent on context). Sometimes used to indicate a block in a block diagonal matrix. When no ambiguity exists the brackets will be omitted, and A_{i} is used

[A]_{i,j} partition of a matrix A. When no ambiguity exists the brackets will be omitted and A_{i,j} is used

A a "\" crossing the letter indicates a diagonal, or block diagonal matrix

\[ A_{1} \oplus ... \oplus A_{k} \] indicates the blockdiagonal matrix,

\[
\begin{bmatrix}
A_{1} & & \\
& & \\
& & \\
& & \\
& 0 & & \\
\end{bmatrix}
\]
Operator Notation

$(..)_R$ real part of a complex expression
$(..)_I$ imaginary part of a complex expression

$\sum_{i=1}^{n} A_i B_i$ summation of the factors $A_i B_i$ for $i$ from 1 to $n$. The summation sign may be deleted when the range is clear from the context; the repeated subscript $i$ then implies the summation

$a + jb$ complex number with real part "a" and imaginary part "b"

$Fo(..)$ Fourier transform of expression between parentheses
$f_X$ Fourier transform of an expression $x$, $f_X = Fo(x)$
The superscript "f" will be omitted, unless ambiguity can occur

$La(..)$ Laplace transform of expression between parentheses
$s_X$ Laplace transform of an expression $x$, $s_X = La(x)$
The superscript "s" will be omitted, unless ambiguity can occur

$Z-t(..)$ Z-transform of expression between parentheses
$z_X$ Z-transform of an expression $x$, $x = Z-t(x)$. The superscript "z" will be omitted, unless ambiguity can occur

$A^t$ transpose of a matrix $A$
$A^*$ complex conjugate of a matrix $A$
$A^{h}$ complex conjugate transpose, or Hermitian transpose, of a matrix $A$

$A^{-1}$ inverse of a matrix $A$
$A^+$ pseudo inverse of a matrix $A$

$|A|$ determinant of matrix $A$
$||{A}||$ vector norm, also the Euclidean length of a vector
\( ||A|| \) matrix norm, unless specified differently, the Frobenius norm is implied ([89], pp. 173)

d/dt derivative with respect to time. For a first and second derivative, a single dot and a double dot over the variable will be used.

\( \Delta t \) time increment

q forward time shift operator, \( q(x_k) = x_{k+1} \)

\( q^{-1} \) backward time shift operator, \( q^{-1}(x_k) = x_{k-1} \)

E(\( . . . \)) expectation of an expression

\( \hat{x} \) a "\(^\wedge\)" over a letter designates an estimate, or an approximation

---

**Greek Alphabet**

\( \omega \) frequency variable (rad/sec)

\( \omega_{ni} \) natural frequency for a mode \( i \), (rad/sec)

\( \Omega_n \) diagonal matrix with natural frequencies for all modes on the diagonal

\( \xi_i \) damping ratio for a mode \( i \)

\( \omega_{di} \) damped natural frequency for mode \( i \), (rad/sec)

\[
\omega_{di} = \omega_{ni} \sqrt{1-\xi_i^2}
\]

\( \Omega_d \) diagonal matrix with damped natural frequencies

\( \lambda_i \) pole value for mode \( i \), (rad/sec)

\( \lambda_i = \alpha_i + j \omega_{di} \)

\( \Lambda \) diagonal matrix with pole values

\( \Lambda_i \) for a repeated mode of multiplicity \( m_i \), a diagonal matrix of dimension \((m_i, m_i)\) with the pole value \( \lambda_i \) on the diagonal
\( \alpha_i \) modal damping factor, (rad/sec)

\[ \alpha_i = -\xi_i \omega_{ni} \]

\( \Lambda \) diagonal matrix with the damping factors for all modes on the diagonal

\( \sigma_x^2 \) covariance of a variable \( x \)

\( \tau \) used to indicate a time variable

\( \delta(t) \) Dirac impulse function

\( \delta_{i,j} \) Kronecker delta,

\[ \delta_{i,j} = 0, \quad i \neq j \quad \text{and} \quad \delta_{i,i} = 1, \quad i = j \]

\( \kappa(A) \) condition number of a matrix \( A \)

---

**Roman Alphabet**

\( C \) damping matrix

\[ \{e\}_i \] unit vector defined as, \( e_{ij} = \delta_{i,j} \)

\( \{e\} \) error vector

\( \{E\} \) error vector

\( G_{X_kY_k} \) estimated power spectrum matrix between \( \{X_k\} \) and \( \{Y_k\} \) at frequency \( \omega_k \)

\( h(t) \) impulse response function

\( H(t) \) matrix of impulse response functions, time domain. When no ambiguity exist, \( H \) is used instead of \( H(t) \)

\( H(s) \) matrix of transfert functions, Laplace domain. When no ambiguity exists, \( H \) is used instead of \( H(s) \)

\( H(\omega) \) matrix of frequency response functions, frequency domain. When no ambiguity exists, \( H \) is used instead of \( H(\omega) \)

\( H(z) \) matrix of impulse response functions, z domain. When no ambiguity exists, \( H \) is used instead of \( H(z) \)
H  orthogonal transformation matrix
I  unity matrix
K  stiffness matrix, also used to indicate an orthogonal transformation matrix
L  matrix with modal participation factors, also used to indicate lower residuals
m_i multiplicity of a repeated mode
M  mass matrix
M_m diagonal matrix with the modal mass for all modes on the diagonal
M_{mi} modal mass, or modal mass matrix, for mode i
N  number of degrees of freedom, number of effective degrees of freedom, number of structure modes
N_c number of complementary modes, number of noise modes, number of computational modes
N_m number of modes, determinable from an estimated model
N_i number of input, or reference, locations, number of sets of initial conditions
N_o number of response locations
N_o' number of principal component responses
N_f number of spectral lines or frequencies
N_t number of time points, equally spaced
r  rank of a matrix
R  upper triangular matrix
R_i matrix with residues for mode i
R_{ci} matrix with combined residues for a repeated mode i
R_{xy}(k) estimated correlation matrix for lag k between the sequences \{x_i\} and \{y_i\}
\( U \) matrix with upper residuals, also used to indicate an upper triangular matrix

\( T \) observation period (sec)

\( V \) matrix with right eigenvectors, or mode shapes

\( V_n \) matrix with real, normal, mode shapes

\( V' \) matrix with left eigenvectors
CHAPTER 1. INTRODUCTION

This dissertation is concerned with the development of some new parameter identification methods for experimental modal analysis of mechanical structures. The identification methods that are presented will estimate directly the parameters in system models that are in the form of differential or finite difference equations, rather than the parameters in the characteristic solutions of such equations. It seems appropriate then to designate these methods as direct parameter model identification methods. Characteristic solutions, and therefore modal parameters, can be calculated after having obtained an estimate for the parameters in the constitutive differential or difference equations.

The theory of these direct parameter model identification methods for multiple input modal analysis of mechanical structures and a study of their applicability will be developed in Chapter 2 and 3 of this dissertation. It is important however to review first in general the existing
methods for experimental modal analysis, and to discuss some of the arguments that have justified the present research effort on new modal parameter identification methods. This is the subject of the first section of this introductory chapter.

In a second section, several existing methods that can be considered as direct parameter model identification methods will be discussed and compared. This section also serves as a context for defining some of the terminology and nomenclature.

As will be argued in Chapter 2, a very important feature of the direct parameter model identification methods that will be developed is that full advantage can be taken of redundant measurement data. In particular, used as a tool for modal analysis, this entails a potential capability to analyze cases with repeated modes. The question may be raised whether such cases are of physical significance and whether such analysis is not merely an academic exercise. Frequently though, cases are encountered where the pole values of several modes occur in one or several clusters. Several modes may be so close that with a given measurement resolution they appear as one mode, a pseudo-repeated mode. An understanding of these latter problems can be achieved through a study of problems with repeated modes. Therefore,
in a third section of this introduction, some theoretical aspects and canonical solutions are presented for problems with repeated modes. Modal parameters will be defined to include such cases.

1.1 General Overview of Experimental Modal Analysis, and Research Contribution

Experimental modal analysis of mechanical structures is a rapidly expanding activity. A variety of test methods, data reduction techniques and parameter estimation methods have been developed over the last three decades for the experimental identification of modal parameters. These methods have been evaluated in a number of survey papers [1-9]. In this section, four technology trees will be used to put most of these methods in perspective, specifically with respect to direct parameter model identification methods that form the subject of this dissertation.

Before discussing the technology trees, some commonly used terminology will be explained. In most survey papers, a distinction is made between modal model identification methods and methods that have been called physical model
methods [2], collocation model methods [5], mathematical input-output model methods [7], and that are in this dissertation further on designated as direct parameter model identification methods. All methods are based on the assumption that the structure is linear and time invariant.

The former methods model the response in terms of the modal parameters, as an expansion into the eigenmodes of the effective degrees of freedom. In other words, the response is described as a superposition of the characteristic solutions of the constitutive differential equation. The model response has the general characteristic of being non-linear in the modal parameters. With some modal model identification methods, modal parameters are determined with such simple procedures as, for example, the half power method to estimate the damping of a mode. Such methods are sometimes qualified as non-statistical [4,5]. With most modal model identification methods, the modal parameters are estimated by fitting the model response to the measured response, either directly or iteratively. Depending on whether modal parameters for one or for several modes can be identified simultaneously, the method is categorized as a single degree of freedom method or a multiple degree of freedom method. With some methods, the modal parameters, specifically the natural frequencies and damping values, are estimated using response data at a number of response locations simulta-
neously, eventually for force input at several force input locations or for several initial conditions. Such estimates of modal parameters are called *global estimates* and the methods are further qualified as global modal model identification methods.

On the other hand, the direct parameter model identification methods model the response directly using the constitutive differential equations. In a sense, these methods address themselves to the mass, stiffness and damping formulation. The parameters in these equations are identified from fitting the model response to the measured response. After identification, such models specify a general input output relation in the form of a multivariate impedance, in which the coefficients in general do not have any physical meaning. The direct parameter model identification methods generally use response data at several response locations simultaneously.

Following Young [2], the direct parameter models can be categorized as *low order complete* and *large order incomplete*. The former models describe the response at a number of response locations with as many modes as are effectively observable in the response; this implies that the number of response locations used in the model is at most equal to the number of modes that are observable. The latter methods describe the response with many more modes
than are observable from the data, that is, the number of response locations used in the model is larger than the number of observable modes. Global estimates for the modal parameters of several modes simultaneously can be calculated as characteristic solutions of the constitutive differential equations after identification of the direct parameter model.

Either of the identification methods can further be qualified by the characteristics of the modal parameters. For the aerospace industry, it is historically been of interest to identify normal mode shapes, i.e. mode shapes with pure real coefficients as encountered for conservative, or proportionally, damped systems. On the other hand, the identification of complex mode shapes, i.e. mode shapes with complex coefficients as in systems with general viscous damping, is found to be valuable, especially to describe the dynamic behavior of more heavily damped structures. Also are the identification methods frequently judged on their capability to estimate modal parameters for highly coupled modes and pseudo-repeated, that is practically repeated, modes. For highly coupled modes, the pole values are such that the contribution of the modes to the response is extremely interwoven, meaning that the response at the resonance frequency of any mode is still severely influenced by the neighboring modes. Repeated modes have theoretically identi-
cal, practically almost identical, pole values. As will be argued in Chapter 2, only repeated modes that have a full complement of independent mode shapes are observable from response data; this condition is therefore further on implied whenever the term repeated mode is used. It will also be demonstrated that the identification of repeated modes requires the analysis of response data for several force input locations or several sets of initial conditions. In general, the identification of repeated modes also requires elaborate postprocessing, unless response data for several force input locations or sets of initial conditions can be analyzed simultaneously.

Following Alleman [7], a primary categorization of modal parameter identification methods can be based on the test method that is used to obtain the vibration data. Four relevant categories can be identified, shown in a first technology tree, Figure 1.1.

![Diagram](image_url)

**Figure 1.1** Test Data Technology Tree
The forced normal mode method is by far the oldest method. Originally developed for ground vibration testing in the aerospace industry, today it still is primarily used for modal surveys of airplane and spacecraft structures. The method is based on tuning the normal modes of the structure, one at a time, by appropriation of the force input at several locations on the structure. Having tuned a mode, the natural frequency, damping value and the normal mode shape, are readily derivable from the data. Many references on this method can be found in the survey papers by Natke [5] and Allemang [7].

A number of modal parameter identification methods have been developed to analyze forced response data, with the forces not measured or not measurable. Such methods are applicable for the analysis of ambient vibration data, caused by wind excitation, traffic vibrations or micro-seismic excitation, in general for the analysis of vibration data under certain operating conditions. Although the input forces are unknown, it is assumed that they are conforming with the statistical model of uncorrelated random normal distributed noise. The methods are further categorized in the technology tree shown in Figure 1.2. Two of the categories, namely the non-statistical stochastic process method and the statistical stochastic process method are as suggested by Gersch [4].
Figure 1.2 Forced Response Data Technology Tree, Forces not Measured

The former category refers to correlation and power spectral density analysis methods. For one degree of freedom systems, for example, the natural frequency and damping value may be estimated as the reciprocal of the time interval for one cycle and the logarithmic decrement respectively of the autocorrelation function. Alternatively, the peak of the power spectral density function identifies the natural frequency, the half power method can be used for a damping estimate and the ratio of the amplitude spectra at the natural frequency to the amplitude at a fixed location gives a mode shape estimate. The non-statistical stochastic process methods could alternatively have been designated non-statistical modal model methods.

Statistical stochastic process methods refer to the estimation of the parameters in an assumed univariate discrete time series model. The time series model is in the form of
an autoregressive moving-average finite difference equation, with as input white noise. This model can also be interpreted as the discrete time representation of the constitutive differential equation for the response, and therefore represents a direct parameter model. Both the maximum likelihood method and a two stage least squares method have been used to identify the parameters and their statistical reliability. Pole values, mode shape coefficients and relative modal participation factors, and their statistical confidence regions can be calculated from the identified parameters in the time series model. Stochastic statistical process methods for vibration data analysis have primarily been developed and used by Gersch [49-57] and Pandit [58-63], and are discussed in more detail in Section 1.2.3.

A third category mentioned in Figure 1.2 is the random decrement method. With this method free decay responses can be estimated from the vibration data. The method was introduced by Cole [10] for the single input case and extended by Ibrahim [32,34] for the multiple input case. It is been used primarily for flight data analysis. The derived free decay responses can be analyzed for modal parameters by any of the methods explained in the proceeding.

In the last decade, a lot of effort has been spent to develop modal parameter estimation methods for forced
response data with measured force inputs. A variety of force input signals can be used, that in general do excite a number of modes simultaneously, as compared to only a single mode for the forced normal mode method. Since force inputs are measured, it is possible to identify completely an input output model of the structure, that consecutively can be used for applications such as response prediction, structural dynamics modification and sensitivity analysis. Free decay response data, that is the response following certain initial conditions, can also be analyzed with similar methods as used for the analysis of forced response data with measured force inputs. The methods will be categorized in two technology trees, shown in Figure 1.3 and 1.4, depending on whether frequency or time is the independent variable.

The formulation of the fast Fourier transform algorithm and the advent of dedicated mini-computer based analysis systems have made the frequency domain modal parameter estimation methods very attractive.
Figure 1.3 Frequency Domain Forced Response Data, Technology Tree

As indicated in Figure 1.3, it is essential for many methods to first have estimates of the frequency response functions. The direct estimation method for frequency response functions, based on the minimization of the squared prediction error in a first order linear regression of output on input, is well established and documented for the single input single output case [11-18]. More recently, the applicability of estimating frequency response functions from response data for simultaneous multiple inputs is been examined, primarily to improve the quality and consistency of the estimated functions [19-22]. The simultaneous estimation of frequency response functions for several inputs is also faster than the estimation of frequency response functions for each of the inputs individually. An other recent development, due to Mitchell [23,24], is the inverse estimation
method for frequency response functions. With this method the inverse of the frequency response function is estimated, based on the minimization of the squared prediction error of a first order linear regression of input on output. Frequency response functions estimated with the inverse method are claimed to approach the true value of the function better in the neighborhood of the resonances, as compared to the functions estimated with the direct method. The opposite situation is observed in the neighborhood of the anti-resonances.

A number of modal model identification methods exist for estimating modal parameters from frequency response functions [13-17,25,26,39]. Very popular are non-statistical single degree of freedom methods that identify modal parameters directly from the data; examples are the half power method to obtain damping values of a mode and the Co-Quad method to obtain normal mode shapes. With the circle-fit method the modal parameters of a single mode can be estimated. Normal or complex mode shapes can be identified with this method. The multiple degree of freedom methods estimate modal parameters of several modes simultaneously, sometimes using iterative non-linear least squares procedures. None of the frequency domain modal model identification methods gives global estimates of modal parameters. The identification of repeated modes also requires postprocessing of
several modal parameter sets, obtained from the repeated application of the method to frequency response functions for several reference locations.

Several direct parameter model identification methods have been developed, apart from the methods to be discussed in this dissertation. Link and Vollan [44,45] propose the estimation of a large order incomplete model using simultaneously vibration data from harmonic base excitation on a shake table at several frequencies. Coppolino [48] uses frequency response functions for one reference location to estimate the parameters in a low order complete model. A correlation analysis is used to find the mode shape coefficients at all response locations from the mode shape coefficients at the response locations used in the low order complete model. Both methods give global estimates of pole values and normal modes, and have successfully been used on experimental data. The detection of repeated modes would require a repeated analysis of vibration data from several base excitation configurations, or of frequency response functions for several reference locations. In earlier work [76-78], I have examined the use of a low order complete model in which the parameters are estimated from frequency response functions, eventually for several reference locations simultaneously, in the presence of constraints such as for example symmetric matrices. The characteristics of this method have been docu-
mented for analytical test cases [77]; it was demonstrated that by analyzing frequency response functions for several reference locations simultaneously, repeated modes can be identified directly. These three direct parameter model identification methods are discussed in more detail in Section 1.2.

In a recent paper, Richardson and Formenti [27] examine the calculation of modal parameters by fitting a rational scalar polynomial to the frequency response functions. This method can be considered as a direct parameter model identification method, using a low order complete model for every frequency response function separately. Using orthogonal polynomials, the parameters of the denominator polynomial can be estimated using frequency response functions for a number of response locations and eventually several reference locations simultaneously. Global estimates for the pole values are calculated as eigenvalues of the denominator polynomial. Residues, and therefore mode shapes and modal participation factors, for all identified modes are calculated from the coefficients in the numerator polynomial, that is fitted to each frequency response function individually. A separate set of mode shapes and modal participation factors is derived for every reference location. Special postprocessing is required to identify a unique set of mode shapes, and eventual repeated modes.
Finally, the estimated frequency response functions can be transformed back to the time domain, to obtain impulse response functions that are then analyzed using the time domain modal parameter estimation methods explained next.

Figure 1.4 shows a technology tree of modal parameter estimation methods for analysis of free decay response data, impulse response functions and forced response data with measured forces in the time domain.

![Technology Tree Diagram]

**Figure 1.4** Time Domain Forced Response Data, Technology Tree

As indicated in this tree, the existing time domain modal model identification methods are designed to analyze free decay response data and impulse response functions. The time domain model response, expressed in terms of modal parameters, appears as a weighted summation of damped complex
exponentials that is non-linear in the modal parameters. Several methods, such as the Complex Exponential method [28,39,40], the Least Squares Complex Exponential method [39] and the Polyreference method [41-43], use Prony's algorithm [29] to derive a higher order linear finite difference equation. The pole values can be found as eigenvalues of the characteristic polynomial of this finite difference equation. For the first two methods, the parameters in the finite difference equation are scalars. The parameters are eventually estimated using response data at many response locations for several reference locations or initial conditions simultaneously, so that global estimates for the pole values are possible. Repeated modes that have a full complement of independent mode shapes can however not be resolved directly, since the pole values are found as eigenvalues of a scalar polynomial. Residues for all identified modes are found from a second estimation process that is repeated for every response location and every reference location or set of initial conditions. A separate set of mode shapes and modal participation factors is derived for every reference location or set of initial conditions, and special post-processing is required to identify the unique set of mode shapes, and eventual repeated modes. With the Polyreference method, response data at a number of response locations and for several reference locations or initial conditions are analyzed simultaneously to estimate the parameters, that now
are square matrices of dimension equal to the number of reference locations or initial conditions. Global pole values are found as eigenvalues of a matrix polynomial, so that it is possible to identify repeated modes directly. Residues for all identified modes are estimated in a second step, repeated for every response location but using response data for all reference locations or initial conditions simultaneously. A unique set of mode shapes, normal or complex, is identified directly for all reference locations or initial conditions, along with a set of modal participation factors for every reference location and initial condition.

Another time domain modal model identification method has been formulated by Ibrahim [30-34]. This method uses all response data for one reference location or initial condition simultaneously to estimate a matrix parameter in a first order finite difference equation, also derivable using Prony's algorithm [29]. Global pole values and mode shapes are obtained as eigenvalues and eigenvectors of the matrix parameter. A second estimation process is required to estimate values for modal participation factors. The identification of repeated modes requires postprocessing on several sets of modal parameters, obtained from a repeated analysis on response data for several reference locations or initial conditions.
These time domain modal model identification methods are discussed in Appendix A.

Apart from the methods developed as part of this research, no direct parameter model identification methods exist that analyze impulse response functions, free decay response or time domain forced response data with measured forces.

In view of this overview then, the contribution of the present research effort, documented in this dissertation, to the area of parameter estimation methods for experimental modal analysis can be summarized as follows:

1. A frequency domain direct parameter model identification method is developed to analyze general multiple input forced response data with measured forces for a number of response locations simultaneously. The model is low order complete, and the parameters are estimated without enforced constraints such as symmetry. Special provisions are made to handle both the situation where the number of available response locations is smaller than the number modes that are observable from the data, and the common situation that many more response locations are available than there are modes observable in the data. Global estimates for pole values, mode shapes and modal participation factors can be calculated from the direct parameter model. When response data
for several input locations is analyzed simultaneously, then a unique set of mode shapes is obtained for all input locations, and repeated modes are resolved directly. The method can be used to analyze frequency response functions at many response locations and for a number of reference locations simultaneously.

2. A time domain direct parameter model identification method is developed to analyze multiple input forced response data with measured forces for several response location simultaneously. The method can also be used to analyze sets of impulse response functions for several reference locations and free decay response data for several sets of initial conditions. The modal parameters that are obtained from the time domain model have the same characteristics as for the frequency domain model mentioned above. Additionally, the time domain method has the advantage that, by direct processing of sampled force input sequences and response sequences, many of the inherent frequency domain processing errors, such as leakage, can be eliminated. Finally, the solution procedures used in the time domain methods are numerically better conditioned and lend themselves more to optimization for memory use and execution speed in a computer than the corresponding procedures for the frequency domain models. The optimal use of the time domain method will however require a change in data acquisition hardware
and software procedures, that with most currently used analysis equipment are oriented towards frequency domain processing of the data.

3. The study of time domain direct parameter model identification also revealed some enlightening interrelations among the time domain modal model identification methods, discussed in Appendix A. In particular, it will be shown that the time domain modal model identification methods can be considered as a subclass of the time domain direct parameter model identification methods discussed in this dissertation.
1.2 Basic Concepts, Early Work

The steady state forced response \{y\} of a N degrees of freedom linear lumped parameter mechanical structure with viscous damping is described by a set of simultaneous linear, constant coefficient, second order differential equations,

\begin{equation}
M\ddot{y} + C\dot{y} + K\{y\} = \{x\}
\end{equation}

where,

\{x\}, is the force input vector, N elements
\{y\}, is the response vector, N elements
M, is the discrete mass matrix, of dimension \((N,N)\)
C, is the discrete viscous damping matrix, of dimension \((N,N)\)
K, is the discrete stiffness matrix, of dimension \((N,N)\)

Let \{H\}_i represent the impulse response for reference location \(i\), i.e. the system response when the force input is zero except for a Dirac impulse at location \(i\). As will be proved in Section 1.3, \{H\}_i is a weighted superposition of the homogeneous solutions of Equation (1),

\begin{equation}
\{H\}_i = Ve^{\lambda t}\{L\}_i
\end{equation}
where,

\( \Lambda \), is a diagonal matrix of dimension \((2N, 2N)\) with the eigenvalues \( \lambda_i \) of the characteristic polynomial of Equation (1) on the diagonal.

\( V \), is a matrix of dimension \((N, 2N)\) with the corresponding right eigenvectors, i.e. for every \( \lambda_i \) and corresponding eigenvector \( \{V\}_i \),

\[
[\lambda_i^2 M + \lambda_i C + K]\{V\}_i = 0
\]

Note that \( \{V\}_i e^{\lambda_i t} \) is an homogeneous solution of Equation (1).

\( \{L\}_i \), is a vector with \( 2N \) elements, representing the weighting factors for the different modes. These weighting factors will further on be called modal participation factors.

The free decay response, \( \{y\}_i \), for some set of initial conditions, is found as solution of (a subscript \( i \) is used to distinguish different sets of initial conditions),

\[
(3) \quad M\{\dot{y}\}_i + C\{\ddot{y}\}_i + K\{y\}_i = 0,
\]

with at time zero,

\[
\{y\}_i = \{y_0\}_i
\]

\[
\{\dot{y}\}_i = \{\dot{y}_0\}_i
\]

The solution is again, see Section 1.3, a superposition of the homogeneous solutions of Equation (1),
\( (4) \) \( \{y\}_i = Ve^{\Lambda t} \{L\}_i \)

where,

\[ \{L\}_i = \tilde{v}^{-1} \begin{bmatrix} \{\hat{y}_o\}_i \end{bmatrix}, \quad \text{and} \quad \tilde{v} = \begin{bmatrix} v \Lambda \\ v \end{bmatrix} \]

Equations (2) and (4) represent the impulse response and free decay response respectively as a superposition of damped complex exponential terms. These expressions form the basis for the time domain modal model identification methods that take advantage of this expansion into complex exponentials [30-43]. Many of these algorithms will estimate \( V \), \( \{L\}_i \) and \( \Lambda \) using as data impulse response functions or free decay responses for a single reference location or set of initial conditions. Recently Vold formulated a complex exponential algorithm that will use impulse responses for several reference locations, or free decay responses for several sets of initial conditions, simultaneously [41-43].

This algorithm, designated the Polyreference method, is derived in Appendix A. The relation to two other important modal parameter estimation algorithms that make use of the expansion of the response into complex exponentials, namely the Ibrahim Time Domain method [30-34] and the Least Squares Complex Exponential method [39,40], is also discussed in this Appendix.
As mentioned in the preceding, direct parameter model identification methods are concerned with the estimation of the parameters in the constitutive differential equations. For example, in Equation (1) the parameters are the matrices $M$, $C$ and $K$.

The method developed by Link and Vollan [44,45] and the method developed by Coppolino [48], both designed to analyze frequency domain data, will be discussed in detail. The frequency domain direct parameter model identification methods that will be developed in Chapter 2 will include both techniques as special cases.

The use of time series analysis methods, applied for modal parameter identification from forced response data with unknown inputs represents an existing time domain direct parameter model identification method and will be reviewed [49-63]. These methods are appealing, since the sampled time data can be analyzed directly without any preprocessing that requires a transformation to the frequency domain and introduces errors, for example leakage, whose effect can be minimized but never eliminated. Time series analysis methods have also been studied and researched for a variety of other applications, such as forecasting [64,67] and automatic control [68], and a wide collection of algorithms with well defined numerical and statistical properties is currently available.
Finally the estimation of the matrices M, C and K in Equation (1), in the presence of constraints such as symmetry, will be discussed [76-78]. The importance of constraining the matrices is sustained by the general consideration that, of all models suitable for describing a given data set, the most parsimonious model, in terms of parameters to be estimated, should be selected.

1.2.1 Identification of Structural System Parameters

( ISSPA - Link and Vollan [44,45] )

The ideas on frequency domain direct parameter model identification methods have originally been inspired by the work of Link and Vollan [44,45]. A method - designated Identification of Structural System Parameters (ISSPA) - was developed, that identifies a large order incomplete model by fitting effective dynamic matrices $M^{-1}C$ and $M^{-1}K$ to the data. The data can be obtained from sinusoidal vibration tests with base excitation, single or multi-point force excitation. Estimates for natural frequencies, proportional damping values and normal mode shapes are calculated from the dynamic matrices, for all effective modes simultaneously. The method has primarily been applied for vibration
analysis of small spacecraft, as an alternative for the forced normal mode method.

Let the response be measured at $N_0$ locations. For base excitation, the response vector $\{y\}$ can be expressed as the summation of the rigid body displacement $\{w\}$ and a displacement vector $\{r\}$ relative to the base displacement $\{u\}$,

\[(5) \quad \{y\} = \{w\} + \{r\}\]
\[(6) \quad \{w\} = T\{u\}\]

The matrix $T$ in Equation (6) is a geometric transformation matrix relating the base displacement vector $\{u\}$ and the rigid body displacement vector $\{w\}$. Substituting $\{y\}$ from Equation (5) in Equation (1), and premultiplying both sides with $M^{-1}$, one derives,

\[(7) \quad \{\ddot{x}\} + C'\{\dot{r}\} + K'\{r\} = -T\{\ddot{u}\} + M^{-1}\{x\}\]

In this equation, $C'$ stands for $M^{-1}C$ and $K'$ for $M^{-1}K$, both matrices of dimension $(N_0, N_0)$.

From the assumption of harmonic base and force excitation at frequency $\omega_i$ follows,

\[(8) \quad \{r\} = \{R\}e^{j\omega_i t}, \quad \{u\} = \{U\}e^{j\omega_i t}, \quad \{x\} = \{X\}e^{j\omega_i t}\]

Using these expressions in Equation (7) then,

\[(9) \quad [-\omega_i^2 I + j\omega_i C' + K']\{R\}_i = \omega_i^2\{F\}_i\]
In this equation $\{F\}_i$ is defined as,

$$\{F\}_i = T\{U\}_i + \omega_i^{-2}M^{-1}\{X\}_i$$

$\{F\}_i$ is the effective load application vector at frequency $\omega_i$. An analytical mass matrix $M$ is required to calculate the contribution of the force application vector $\{X\}_i$. For pure base excitation, Equation (9) is independent of an analytical mass matrix. $\{R\}_i$ will in general be complex; $\{F\}_i$ is real, by the assumption of sinusoidal excitation.

Equation (9) hold for all frequencies $\omega_i$ at which measurement data is available. Assume $N_f$ frequencies, and $N_f \geq N_o$. Let $R$ and $F$ be matrices of dimension $(N_o, N_f)$ with the response data and the effective loading at the $N_f$ frequencies. Let $\Omega$ be a diagonal matrix of dimension $(N_f, N_f)$ with $\omega_i$ on the diagonal. Equations like Equation (9), for all frequencies, can then be put together in one equation,

$$[-\Omega^2 R^t + j\Omega R^t C^t + R^t K^t] = \Omega^2 F^t$$

Let $A$ be the matrix of dimension $(N_f, N_o)$ with the real part of response matrix $R^t$, and let $B$ be the imaginary part of $R^t$. Equating real and imaginary part of Equation (11) yields two real equations in the unknown matrices $C'$ and $K'$,

$$AK^t - \Omega^2 A - \Omega BC^t = \Omega^2 F^t$$
$$BK^t - \Omega^2 B + \Omega AC^t = 0$$
Designating the pseudo-inverses of \( A \) and \( B \) by \( A^+ \) and \( B^+ \) respectively [46,47], Equation (12) and (13) can be solved a minimum norm least squares estimate of \( K' \) and \( C' \),

\[
(14) \quad K'^t = [A + \Omega B A^+ \Omega^{-1} B]^+ \Omega^2 [A + F^t] + \Omega B A^+ \Omega B \\
(15) \quad C'^t = -B^+ \Omega [A + F^t] - \Omega^{-1} A K'^t
\]

The rank of the identified matrices \( C' \) and \( K' \) will be equal to the rank of the matrices \( A \) and \( B \). The rank of these matrices cannot be more than \( N_o \), assuming that \( N_f \) is larger than \( N_o \). Practically, the rank will equal the number of degrees of freedom, \( K \), that effectively contribute to the response. In general, \( N \) will be smaller than \( N_o \), so that the matrices \( K' \) and \( C' \) will be rank deficient. The identification of rank deficient dynamic matrices is typical for a large order incomplete model.

An estimate of the natural frequencies \( \omega_{ni} \) and the normal mode shapes \( \{ v_n \} \) can be found from the real eigenvalue decomposition of \( K' \), estimated from Equation (14), that is for the associated conservative system. Since \( K' \) will in general be non-symmetric, a left and right eigenvalue decomposition can be considered defined by Equations (16) and (17) respectively,

\[
(16) \quad K' v' = v' \Omega^2_n 
\]
\begin{equation}
(17) \quad \mathbf{K}'\mathbf{V} = \mathbf{V} \sum_{n}^{2}
\end{equation}

\( \mathbf{V}' \) and \( \mathbf{V} \) are the left and right eigenvectors respectively. Left and right eigenvectors can always be normalized such that,

\begin{equation}
(18) \quad \mathbf{V}'^{t}\mathbf{V} = \mathbf{I}
\end{equation}

The right eigenvectors \( \mathbf{V} \) are selected as estimate for the normal mode shapes and are therefore further on designated \( \mathbf{V}_{n} \). The orthogonality of the normal mode shapes can be checked when an analytical mass matrix is available.

Consider now the transformation,

\begin{equation}
(19) \quad \{\mathbf{R}\}_i = \mathbf{V}_{n}\{\mathbf{Q}\}_i
\end{equation}

Substituting \( \{\mathbf{R}\}_i \) from Equation (19) in Equation (9), premultiplying both sides with \( \mathbf{V}'^{t} \) and after simplifying the resulting expression using Equation (17) and (18), one obtains,

\begin{equation}
(20) \quad [\!\!\!\!\!\!\!-\omega_{i}^2\mathbf{I} + j\omega_{i}\mathbf{V}'^{t}\mathbf{C}'\mathbf{V}_{n} + \sum_{n}^{2}\![\!\!\!\!\!\!\!\!]\{\mathbf{Q}\}_i] = \omega_{i}^2\mathbf{V}'^{t}\{\mathbf{F}\}_i
\end{equation}

With \( \mathbf{C}' \), estimated from Equation (15), \( \mathbf{V}'^{t}\mathbf{C}'\mathbf{V}_{n} \) will in general not be diagonal. Treating the coupling terms as secondary effects, i.e. assuming proportional damping, the diagonal elements can be used to obtain approximate values for the damping ratios \( \xi_{i} \),
(21) \((V'^t c' v_n)_{i,i} = -2 \xi_i \omega_{ni}\)

Alternatively, assuming again proportional damping, a diagonal damping matrix \(\mathbf{\xi}'\) can be calculated from Equation (20). Hence, let \(Q\) and \(P\) be matrices of dimension \((N_o, N_r)\) defined by,

(22) \(R = V_n Q\),
(23) \(P = V'^t F\)

Using Equations (22) and (23), the following equation can be derived from Equation (20), assuming \(V'^t c' v_n\) to be a diagonal matrix \(\mathbf{\xi}'\),

(24) \(-\Omega^2 q^t + j \Omega q^t \mathbf{\xi}' + q^t \Omega_n^2 = \Omega^2 p^t\)

From Equation (22) also follows,

(25) \(V'^t R = Q\)
(26) \(A V' = (Q^t)_R\)
(27) \(B V' = (Q^t)_I\)

Equating real and imaginary parts in Equation (24), and substituting \((Q^t)_R\) and \((Q^t)_I\) from Equations (26) and (27),

(28) \(-\Omega^2 A V' - \Omega B V' \mathbf{\xi}' + A V' \Omega_n^2 = \Omega^2 P^t\)
(29) \(-\Omega^2 B V' + \Omega A V' \mathbf{\xi}' + B V' \Omega_n^2 = 0\)

The only unknowns in Equation (28) and (29) are the diagonal elements \(c_i\) of \(\mathbf{\xi}'\). A least squares estimate for every

31
element \( c_1 \) can be found. For example, using Equation (28),

\[
(30) \quad \{\Omega BV'\}_i c_i = \{AV'\}_i \omega_{ni}^2 - \{\Omega^2 AV'\}_i - \{\Omega^2 pt\}_i
\]
or,

\[
(31) \quad \{a\} c_i = \{b\}
\]
The least squares estimate for \( c_i \) equals,

\[
(32) \quad c_i = \{a\}^t \{b\} (\{a\}^t \{a\})^{-1}
\]
The damping ratio \( \xi_i \) then follows from,

\[
(33) \quad c_i = -2\xi_i \omega_{ni}
\]

Finally, a verification of the estimated values is obtained by recalculation of the dynamic response.

The essential characteristics of the ISSPA method can be summarized as follows:

1. An estimation scheme is set up to estimate the effective dynamic matrices \( M^{-1}C \) and \( M^{-1}K \) by simultaneous analysis of response data at \( N_o \) response locations. A minimum norm pseudo-inverse procedure allows the matrices to be estimated even when the number of effective modes in the response data is less than the number of response locations.

2. An estimate of the natural frequencies and normal modes is obtained from a real eigenvalue decomposition of \( M^{-1}K \).
The orthogonality of the normal modes is not explicitly used in the identification method but can be checked when an analytical mass matrix is available.

3. Alternative procedures have been examined to estimate the proportional damping values.

4. The model is essentially a large order incomplete model; many more modes may be calculated from the eigenvalue decomposition of $M^{-1}K$ than actually are contributing to the response. Procedures for selecting computational modes from the physical meaningful modes have not been documented. The method also implies that at least as many response locations are available as there are modes effectively contributing to the response in the examined frequency range.

5. The method does not estimate any elements in the mass matrix. Consequently, no experimental information is obtained on the modal participation factors of the identified modes. Modal participation factors can however be calculated using an analytical mass matrix and the estimated normal mode shapes.
1.2.2 Simultaneous Frequency Domain Technique

(SFD - Coppolino [48])

In a recent paper [48] Coppolino describes a direct parameter model identification method for estimating modal parameters from frequency response functions, designated Simultaneous Frequency Domain technique (SFD). The SFD technique is based on the concept that the characteristics of a structural dynamics system can be described in terms of an abbreviated set of physical degrees of freedom when interest is restricted to a frequency band containing a limited number of modes. Essentially a low order complete model is identified by fitting effective dynamic matrices $M^{-1}C$ and $M^{-1}K$ to the frequency response functions for the abbreviated set of degrees of freedom. Estimates of natural frequencies, damping values, and normal or complex mode shapes are calculated from the effective matrices. The SFD technique has primarily been applied for modal analysis of aerospace structures. The technique has been compared with some commercially available frequency domain modal model identification methods, and been found to give comparable results.

Assume that frequency response functions are measured for $N_o$ response locations, over a frequency range in which $N$ modes effectively contribute to the system response, and $N \ll N_o$. 

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An abbreviated set of frequency response functions at $N_{01}$ response locations, $N_{01}$ approximately equal to but larger than $N$, are considered as independent degrees of freedom and are used to estimate the parameters in an effective dynamic equation that describes the response characteristics of the system. The frequency response functions at the remainder $N_{02}$ response locations are considered as the dependent degrees of freedom. A fixed dependence relation between the dependent and independent degrees of freedom is assumed and estimated from a correlation analysis.

Let $\{Y_1\}_i$ represent a vector of $N_{01}$ elements with the values for the frequency response functions at frequency $\omega_i$ for the $N_{01}$ independent degrees of freedom. The frequency response functions are with respect to some reference location that is not necessarily one of the response locations. The technique as formulated by Coppolino uses frequency response functions for one reference location; the extension of the method to include functions for several reference locations is possible and will be discussed in Chapter 2. An effective dynamic equation can be written as,

\begin{equation}
-\omega_i^2 \{Y_1\}_i + j\omega_i C' \{Y_1\}_i + K' \{Y_1\}_i = \{Q\}
\end{equation}

In this equation, $C'$ stands for $M^{-1}C$ and $K'$ for $M^{-1}K$. $C'$ and $K'$ are effective dynamic matrices of dimension $(N_{01}, N_{01})$. The vector $\{Q\}$ is a vector of $N_{01}$ elements, that represent
the equivalent force application at the \( N_{o1} \) degrees of freedom for the given reference location.

Let \( \{Y_2\}_i \) be a vector of \( N_{o2} \) elements, with the values for the frequency response functions at the \( N_{o2} \) dependent degrees of freedom. A linear dependence relation between \( \{Y_1\}_i \) and \( \{Y_2\}_i \) can be expressed as,

\[
(35) \quad \{Y_2\} = T\{Y_1\}
\]

In this equation \( T \) is a matrix of dimension \( (N_{o2}, N_{o1}) \) which can be real or complex.

Values for the frequency response functions at \( N_f \) frequencies \( \omega_i \) can be used simultaneously to estimate \( C', K', \{Q\} \) and \( T \) in Equations (34) and (35). Here, let \( Y_1 \) and \( Y_2 \) be matrices of dimension \( (N_{o1}, N_f) \) and \( (N_{o2}, N_f) \) respectively, with the values of the frequency response functions at the dependent and independent degrees of freedom. Let \( \Omega \) be a diagonal matrix of dimension \( (N_f, N_f) \) with the frequency values \( \omega_i \) on the diagonal, and let \( [u] \) be a row vector of \( N_f \) elements with every element equal to one. Equation (34) represents for all frequencies \( \omega_i, i = 1 \ldots N_f \), a set of linear equations which may be written in matrix form as,

\[
(36) \quad \begin{bmatrix} C', K', \{Q\} \end{bmatrix} \begin{bmatrix} jY_1 \Omega \\ Y_1 \\ -[u] \end{bmatrix} = [Y_1 \Omega^2]
\]
Since \( C', K', \{Q\} \) are real, the following real equation can be derived by equating real and imaginary parts in Equation (36),

\[
\begin{pmatrix}
C', K', \{Q\} & -((Y_1)_I) \Omega & (Y_1)_R \Omega \\
\end{pmatrix} \begin{pmatrix}
(Y_1)_I \Omega \\
(Y_1)_R \Omega \\
\end{pmatrix} = \begin{pmatrix}
(Y_1)_R \Omega^2 & (Y_1)_I \Omega^2 \\
(Y_1)_R & (Y_1)_I \\
-u_j & 0 \\
\end{pmatrix}
\]

or,

(38) \( XA = B \)

The least squares solution for \( X \) in Equation (38) equals,

(39) \( X = [BA^t][AA^t]^{-1} \)

A complex matrix \( T \) can be estimated similarly, from a correlation analysis between \( Y_2 \) and \( Y_1 \),

(40) \( Y_2 = TY_1 \)

(41) \( T = [Y_2 Y_1^t][Y_1 Y_1^t]^{-1} \)

To estimate a real correlation matrix \( T \), a real equation is first derived from Equation (40) by equating real and imaginary parts and assuming \( T \) to be real,

(42) \( \begin{pmatrix}
(Y_2)_R \\
(Y_2)_I \\
\end{pmatrix} = T \begin{pmatrix}
(Y_1)_R \\
(Y_1)_I \\
\end{pmatrix} \)

or,

(43) \( B = TA \)
A least squares estimate is given by,

\[(44) \quad T = [BA^t][AA^t]^{-1}\]

All modal parameters can be calculated from the estimated \(C', K', \{Q\}\) and \(T\). Damping ratios, natural frequencies and complex mode shapes follow from the complex eigenvalue decomposition,

\[(45) \quad \begin{bmatrix} -C' & -K' \\ I & 0 \end{bmatrix} \begin{bmatrix} V_1 \Lambda \\ V_1 \end{bmatrix} = \begin{bmatrix} V_1 \Lambda \\ V_1 \end{bmatrix} \Lambda\]

\(\Lambda\) is a the diagonal matrix of dimension \(2N_0, 2N_0\) with the pole values \(\lambda_i\) and \(V_1\) is a matrix of dimension \((N_0, 2N_0)\) with the corresponding complex mode shapes. Damping ratios \(\xi_i\) and natural frequencies \(\omega_{n_i}\) are related to the pole values by Equation \((46)\),

\[(46) \quad \lambda_i = -\xi_i \omega_{n_i} \pm j\omega_{n_i} \sqrt{1-\xi_i^2}\]

A modal parameter expansion for \(\{Y_i\}\) is given by Equation \((47)\) through \((49)\). The residues can be calculated using Equation \((50)\).

\[(47) \quad \{Y_1\}_i = V_1 [j\omega_i I - \Lambda]^{-1}\{L\}\]

with,

\[(48) \quad \{L\} = \begin{bmatrix} V_1 \Lambda \end{bmatrix}^{-1}\{Q\} \begin{bmatrix} 0 \\ V_1 \end{bmatrix}\]

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\[ (49) \quad \{Y_1\}_i = \frac{\sum_{k=1}^{2N_{o1}} \{R_k\}}{(j\omega_i - \lambda_k)} \]

with,

\[ (50) \quad \{R_k\} = \{V_1\}_k \]

These expressions show that the estimated vector \{Q\} allows the calculation of the modal participation factors and consequently the calculation of the residues.

Let \( V_2 \) be the matrix of dimension \( (N_{o2}, 2N_{o1}) \) with the mode shapes coefficients for the dependent degrees of freedom. \( V_2 \) can be calculated from Equation (47) and the following equations,

\[ (51) \quad \{Y_2\}_i = T\{Y_1\}_i \]
\[ (52) \quad \{Y_2\}_i = V_2[j\omega_i I - \Lambda]^{-1}\{L\} \]

Substituting \( \{Y_1\}_i \) from Equation (47) in Equation (51) and comparing the resulting expression for \( \{Y_2\}_i \) with the one from Equation (52) yields,

\[ (53) \quad V_2 = TV_1 \]

The matrix \( T \) in above equations is as calculated from Equation (41).

Natural frequencies and normal mode shapes can also be calculated from the real eigenvalue decomposition of the
associated conservative system,

\[ (54) \quad K' v_{n1} = v_{n1} \Omega_n^2 \]

\( \Omega_n \) is a diagonal matrix with the natural frequencies and \( v_{n1} \) is the matrix of normal modes. An estimate for the proportional damping ratios follows from the diagonal elements in \( v_{n1}^{-1} C' v_{n1} \),

\[ (55) \quad (v_{n1}^{-1} C' v_{n1})_{i,i} = -2 \xi_i \omega_{ni} \]

Finally the matrix \( v_{n2} \) with the components of the normal modes at the dependent degrees of freedom follows from,

\[ (56) \quad v_{n2} = T v_{n1} \]

in which \( T \) is the real matrix estimated from Equation (43).

A confidence check of the estimated values is carried out using a correlation coefficient between the recalculated response and the measured response.

The principal characteristics of the SFD technique can be summarized as follows:

1. A subset of frequency response functions at \( N_{o1} \) locations, with \( N_{o1} \) close to the expected number of effective degrees of freedom, is selected and used to estimate the effective dynamic matrices \( M^{-1}C \) and \( M^{-1}K \) of dimension
\((N_{01}, N_{01})\). The frequency response functions at all other locations are assumed to be dependent and linearly related to the frequency response functions in the selected subset. This avoids the rank loss that potentially occurs in the equation system to be solved for the dynamic parameters, especially when the total number of response locations is much larger than the number of effective degrees of freedom. The disadvantage is that the estimates of the modal parameters are dependent on the selected subset of frequency response functions. To increase confidence in the extracted modal parameters, the analysis should be repeated for several subsets, especially when local and appendage modes are expected. Also, it is essential that the total number of available frequency response functions equals at least the expected number of effective degrees of freedom.

2. Global estimates of pole values and mode shapes are identified from the effective dynamic matrices. In general, complex mode shapes are calculated; but, as with the ISSPA method, it is possible to use the dynamic matrices to calculate normal mode shapes and proportional damping values.

3. The SFD technique also estimates a vector \(\{Q\}\), whose elements represent the equivalent force application. The vector \(\{Q\}\) can be used to calculate the proper contribution of the different modes to the response, that is the modal participation factors.
1.2.3 The Use of Time Series Analysis for Structural Parameter Identification.

(Gersch [49-57], Pandit [58-63])

The applicability of time series analysis techniques to sampled response data of mechanical structures under the assumption of random excitation has been examined by Gersch and Pandit. Gersch has used the techniques primarily for the analysis of response data taken on multi-story buildings under ambient wind and micro-seismic excitation. The application areas examined by Pandit include the analysis of vibration records of machine tools for chatter analysis, the modelling of control systems and also forecasting and reliability studies. From the results obtained by both researchers, the time series analysis techniques have shown to be a viable method for estimating power spectra without frequency domain processing and for estimating the natural frequencies, damping factors, mode shapes and relative modal participation factors.

Let a $N$ degree of freedom lumped parameter mechanical structure, described by Equation (1), be excited by random uncorrelated forces, that are not measured. Let the response at a specific location be regularly sampled with a time increment $\Delta t$, and be represented by the sequence $y_k$, $k = 0 \ldots N_t$. The response has a rational spectral density,
and the sequence $y_k$ can therefore be described by the mixed autoregressive moving-average model,

$$a_0 y_k + a_1 y_{k-1} + \cdots + a_{2N} y_{k-2N} = b_0 x_k + b_1 x_{k-1} + \cdots + b_{2N} x_{k-2N}$$

The sequence $y_k$ is assumed to represent the true response, contaminated with random uncorrelated noise. The sequence $x_k$, used to model the unknown excitation signal, is a sequence of random uncorrelated normally distributed values. The statistics of this sequence are completely described by the first and second moments,

$$E(x_k) = 0$$
$$E(x_k x_k) = \sigma_x^2$$
$$E(x_k x_{k+i}) = 0, \ i \neq 0$$

In Equation (57), it can always be assumed that $a_0$ and $b_0$ are both equal to 1, since the variance $\sigma_x^2$ is essentially unknown.

The most general method of identifying the unknown parameters $a_i$, $b_i$ and $\sigma_x^2$ in Equation (57) is based on determining the maximum of a properly defined likelihood function using non-linear optimization algorithms. Both Pandit and Gersch, in his earlier work [49-52], have used this approach; a likelihood function is selected that is proportional to $\sigma_x^2$ and optimization algorithms are used that iteratively
identify the parameters $a_i$ and $b_i$ that minimize $\sum_{x^*}^2$.

To reduce the severe computational effort required to maximize the likelihood function, linear estimation methods have been suggested. In particular, the applicability of a solution procedure that consists of a sequential solution of two ordinary least squares problems (in control theory this procedure is known as a two stage least squares method [65, 66]) has been examined by Gersch [53-55]. The underlying ideas of this two stage least squares procedure will briefly be outlined.

The autoregressive moving-average model, Equation (57), can alternatively be represented by a pure autoregressive model as in Equation (61) or a pure moving-average model as in Equation (62) ([67], pp. 46-48),

\[ \sum_{i=0}^{\infty} \varphi_i y_{k-i} = x_k \quad (61) \]

\[ y_k = \sum_{i=0}^{\infty} h_i x_{k-i} \quad (62) \]

Note that Equation (62) is simply a discrete time representation of the response of a linear system as the convolution of impulse response sequence $h_i$ with the input sequence $x_i$.

In a first step of the procedure, the infinite order autoregressive model described by Equation (61) will be truncated to a finite order $p$,
(63) \[ y_k = - \sum_{i=1}^{p} g_i y_{k-i} + x_k \]

The prediction error \( x_k \) is assumed to be uncorrelated and random, a reasonable assumption when \( p \) is sufficient larger than \( 2N \). A least squares estimate can be found for the coefficients \( g_i, \) \( \hat{\varepsilon}_i, \) by solving simultaneously equations like Equation (69) for all available \( k \).

Consecutively the prediction errors \( \hat{x}_k \) can be calculated by a backsubstitution,

(64) \[ \hat{x}_k = y_k + \sum_{i=1}^{p} \hat{g}_i y_{k-i} \]

The sequence \( \hat{x}_k \) is used as an estimate for the random uncorrelated noise sequence that had generated the sequence \( y_k \). An estimate for \( \sigma_x^2 \) is given by,

(65) \[ \sigma_x^2 = (1/N_t) \sum_{k=0}^{N_t} x_k^2 \]

Having obtained an estimate for the random noise sequence \( \hat{x}_k \) as the prediction errors of the autoregressive model defined by Equation (61), a least squares estimate for the coefficients \( a_i \) and \( b_i \) in the autoregressive moving-average model defined by Equation (57), can be found, as a second step, by solving simultaneously the equations like Equation (57) for all available \( k \).

A critical step in the procedure is assigning the value of \( p \)
in the autoregressive model. The adequacy of the selected order \( p \) can be verified by checking how well the prediction error sequence \( \hat{x}_k \) is a sequence of random uncorrelated values. The solution of the second least squares problem also requires the fixing of the order \( 2N \) of the autoregressive moving-average model. Overestimating the order will damage the parsimony of the model and increase the variance of the estimated parameters. An underestimation of the order, on the other hand, will generate a model that does not properly fit the observed data, and can not be used for prediction of structure response or calculation of structural parameters. Fortunately, several techniques are becoming available that can be used to construct an intelligent strategy for optimal selection of the order to be used in the autoregressive moving-average model.

Gersch has developed a particular realization for this two stage least squares procedure that only uses the sampled correlation function \( r_{yy}(k) \) of the observed responses to solve both least squares equation systems [54-56]. The selection of the optimal order of the model is based on the minimization of Akaike's Information Criterion [69-75], and the procedure is also claimed to be extendable for the multivariate case.

The autoregressive part of the estimated autoregressive moving-average model can be used to calculate the natural
frequencies and damping ratios. They are related to the poles \( z_i \) of the polynomial,

\[
(66) \quad z^2N + a_1z^{2N-1} + \ldots + a_{2N} = 0
\]

by following equations,

\[
(67) \quad z_i = e^{\lambda_i \Delta t} \\
(68) \quad \lambda_i = -\xi_i \omega_{ni} \pm j\omega_{ni} \sqrt{1-\xi_i^2}
\]

The relative contribution of the different modes to the response can be calculated from the estimated coefficients in the moving-average part of the model. Hence mode shapes can be obtained when the analysis is done on response data, that was taken at several response locations of the structure.

As already mentioned in the beginning of this section, the estimated autoregressive moving-average model can also be used to estimate power spectra [49-51,60,70]. This parametric approach is observed to be far more accurate than the commonly used non-parametric point wise estimation of power spectra using frequency domain processing techniques that are based on the Discrete Fast Fourier Transform algorithm. The accuracy of the latter method is completely determined by the particular data acquisition and windowing procedure that is used to minimize the errors associated with the frequency domain processing of the data.
The techniques have originally been applied to analyze response data under the assumption of stationarity. Recently Gersch has argued that the assumption of stationarity can be relaxed and has demonstrated the feasibility of identifying structural parameters from time series analysis of non-stationary response data [56,57].

1.2.4 Direct Structure Parameter Identification in the Presence of Constraints (DSPI - Leuridan [76-78])

A key consideration in estimation theory for selecting a model to describe a set of observed data is parsimony. Of all available models, the model that is most parsimonious in terms of parameters to be estimated should be used and may be expected to give the estimates of the parameters with smallest variance.

In view of this consideration, a frequency domain direct parameter model identification method has been developed [77,78] that estimates the mass, stiffness and damping matrices in the presence of constraints. The method uses a low order complete model, in the form of Equation (1), in which the parameters are estimated by simultaneous analysis
of multiple input, multiple output data, eventually reduced to frequency response functions.

In particular, the ability to constrain the matrices to be symmetric is of considerable practical importance. It allows a model, that satisfies the Maxwell-Betti reciprocity principle, to be used to fit the data. The symmetric matrices in this model can consecutively be used to calculate directly, natural frequencies, damping ratios, mode shapes, and also the modal mass for all modes.

The discussion in this section will be restricted to the some general conditions for which a constrained model can be identified.

Let the structure be measured at \( N_o \) response locations for simultaneous input at \( N_i \) locations that are assumed to be a subset of the \( N_o \) response locations. The sequence of the response locations can be selected so that the first \( N_i \) response locations correspond to the locations where force is applied. Let \( \{y\} \) be the response vector, and let \( \{x\} \) be the force input vector. Let the matrices \( M \), \( C \) and \( K \) in the lumped parameter model, Equation (1), be constrained and let the constrained matrices be designated by \( M_c \), \( C_c \) and \( K_c \) respectively. As mentioned, the DSPI method involves a model for frequency domain data. After transforming Equation (1) to the frequency domain, Equation (69) is derived, for
example for $N_f$ frequencies. The value of the Fourier transform of $\{x\}$ and $\{y\}$ is represented by $\{X\}_i$ and $\{Y\}_i$, 

$$i = 1 \ldots N_f.$$

(69)  

$$-\omega_i^2 M_0 \{x\}_i + \boldsymbol{3}\omega_i C_0 \{Y\}_i + K_0 \{Y\}_i = \{X\}_i \quad i = 1 \ldots N_f$$

Let $S$ be any matrix with the property,

(70)  

$$S \{X\}_i = \{X\}_i \quad i = 1 \ldots N_f$$

Let $\{X_1\}_i$ represent the partition of $\{X\}_i$ with the first $N_1$ elements, and $\{X_2\}_i$ the partition with the remainder elements, that are all equal to 0. Conforming to this partitioning, the matrix $S$ can be partitioned as,

(71)  

$$S = \begin{bmatrix}
S_{1,1} & S_{1,2} \\
S_{2,1} & S_{2,2}
\end{bmatrix}$$

Two non-trivial equations follow from Equation (70),

(72)  

$$S_{1,1} \{X_1\}_i = \{X_1\}_i \quad i = 1 \ldots N_f$$

(73)  

$$S_{2,1} \{X_1\}_i = 0 \quad i = 1 \ldots N_f$$

or,

(74)  

$$S_{1,1} X_1 = X_1$$

(75)  

$$S_{2,1} X_1 = 0$$

In Equation (74), $X_1$ is a matrix with $\{X_1\}_i$ as columns.
Assume that a non-trivial matrix \( S \), with the property expressed by Equation (70), can be found. Premultiplying both sides of Equation (69) with \( S \), and making use of this property, then yields,

\[
(76) \quad -\omega_i^2 M'_o \{y\}_i + j\omega_i C'_o \{y\}_i + K'_o \{y\}_i = \{x\}_i
\]

In this equation \( M'_o \) equals \( SM_o \), and similar for \( C'_o \) and \( K'_o \). When \( M'_o \), \( C'_o \) and \( K'_o \) satisfy the same constraints as \( M_o \), \( C_o \) and \( K_o \), then Equation (76) defines a constrained model that is equally valid as the model defined by Equation (69). For the model, specifically the parameters \( M_o \), \( C_o \) and \( K_o \), to be uniquely identifiable from the data, it is required that the only non-trivial matrix \( S \) equals \( I \). This will require specific constraints on the matrices \( M_o \), \( C_o \) and \( K_o \) in Equation (69).

The matrices \( M_o \), \( C_o \) and \( K_o \) can reasonably be assumed to be real. Since in general \( S \) can be complex, this means that,

\[
(77) \quad (S)_I \begin{bmatrix} M_o \\ C_o \\ K_o \end{bmatrix} = 0
\]

It will be assumed that the coefficient matrix in Equation (77) is of rank \( N_o \), so that only \( (S)_I \) equal to 0 can be a solution. This assumption is justified for a low order complete model in which \( \{y\} \) represents the response of an \( N_o \)
degree of freedom system, in which case $M_c$ is necessarily full rank. Letting $S$ be real, Equation (74) becomes,

\[(78) \quad S_{1,1} [(X_1)_R (X_1)_I] = [(X_1)_R (X_1)_I] \]

When the matrix $[(X_1)_R (X_1)_I]$ is full rank, then the only possibility for $S_{1,1}$ in Equation (78) is $I$. Otherwise, $S_{1,1}$ is essentially arbitrarily. Under the same condition, $S_{2,1}$ will equal 0, or be arbitrarily. In either case however, $S_{1,2}$ and $S_{2,2}$ will be arbitrarily. In conclusion, with the constraint of real matrices, the model expressed by Equation (69) is uniquely identifiable from the data when $N_1$ equals $N_0$ and $[(X_1)_R (X_1)_I]$ is full rank.

Without adding additional constraints, the model can only be made identifiable by factoring out any of the matrices, $M_c$, $C_c$ or $K_c$. Factoring out $M_c$, a model of the form used in the ISSPA method, Equation (7), and in the SFD method, Equation (34), is derived. The conditions for which the parameters in such a model are uniquely defined by the data will be discussed in Chapter 2 for the general multiple input, multiple output case.

With the DSPI method, the matrices are additionally constrained to be symmetric and only elements on the diagonal and a selected number of upper diagonal bands are estimated. Eventually, an assumed value may be assigned to elements in
any location within the bands. Interestingly, under the constraint of symmetry, the matrices are uniquely defined by the data, even when the matrix \( [(X_1)_R (X_1)_I] \) is not full rank [77].

The characteristics of the DSPI method have primarily been examined on analytical data sets. As mentioned before, it was demonstrated that repeated poles can be identified directly, when response data for multiple force inputs is analyzed simultaneously.

With this method, fewer parameters are estimated on a given data set, so that the variance of the estimates is expected to be smaller. This is however accompanied by an increased complexity of the estimation scheme and consequently, for a given data set to be analyzed, an increased cost in computation.
1.3 Canonical Response Expressions and General Definition of Modal Parameters

This section deals with a general concise derivation of the canonical free decay and forced response expressions of a linear viscous damped mechanical structure, and the definition of the modal parameters from these expressions. The equations of motion of such a system are a set of linear, constant coefficients, second order differential equations. The coefficients in this set of equations can be grouped together to form mass, stiffness and damping matrices, that will be assumed to be real but not necessarily symmetric. The reason for being concerned with the non-symmetric case is that these matrices, when estimated on experimental data, will in general not come out to be symmetric, unless constraints in the estimation procedure enforce the symmetry condition. Modal parameters of a structure will be defined for the general case, including the possibility of repeated modes that have a full complement of independent mode shapes. As will be argued in Chapter 2, this is really all what can be observed from experimental data. Simplifications and additional properties that exist when the matrices have
a special structure, for example they are symmetric, will be discussed. For further discussion, Lancaster [79], Meirovitch [80] and Potter [81] are suggested.

1.3.1 Canonical Response Expressions

Equation (1) represents the equation of motion of a \( N \) degrees of freedom, linear, lumped parameter mechanical structure with viscous damping, described by the coordinates \( \{ y \} \).

\[
(1) \quad M \ddot{\{ y \}} + C \dot{\{ y \}} + K \{ y \} = \{ x \}
\]

The vectors \( \{ x \} \) and \( \{ y \} \) represent the force input and response respectively and are function of time. \( M, C \) and \( K \) are real matrices of dimension \( (N,N) \). The \( N \) second order differential equations represented by Equation (1) can be linearized,

\[
(2) \quad \begin{cases} \{ \ddot{y} \} \\ \{ \dot{y} \} \end{cases}, \quad A = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}, \quad B = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}, \quad \{ \ddot{x} \} = \begin{cases} 0 \\ \{ \dot{x} \} \end{cases}
\]

\[
(3) \quad A \{ \ddot{y} \} + B \{ \dot{y} \} = \{ \ddot{x} \}
\]

A set of \( 2N \) first order differential equations is derived, also known in literature as Hamilton's canonical equations. When the original matrices are symmetric, then so will be \( A \)
and B. Assume now that A is invertible,

\[
A^{-1} = \begin{bmatrix} M^{-1}CM^{-1} & M^{-1} \\ N^{-1} & 0 \end{bmatrix}
\]

It is worth noting that \(A^{-1}\) exists when \(M^{-1}\) exists. This requires \(|M|\) to be non-zero. However \(|M|\) is the highest order coefficient of the characteristic equation corresponding to Equation (1). When \(|M|\) equals zero, then the order will be less than \(2N\) which implies that less than \(2N\) eigenvalues exist. This entails that the coordinates \(\{y\}\) describe a system with less than \(N\) degrees of freedom, which indicates the existence of dependence relations within the coordinates \(\{y\}\). Hence, requiring that Equation (1) describes a \(N\) degree of freedom system is a sufficient condition for \(M^{-1}\) to exist. Strictly stated then, \(\{y\}\) should be termed generalized coordinates. After factoring out \(A^{-1}\), Equation (3) becomes,

\[
\ddot{\{y\}} + A^{-1}B\{\dot{y}\} = A^{-1}\{x\}
\]

with,

\[
A^{-1}B = \begin{bmatrix} M^{-1}C & M^{-1}K \\ -I & 0 \end{bmatrix} = -D
\]

\[
P = \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix}
\]

Equation (5) can then be written as,
(7) \[ \tilde{y} - D\tilde{y} = P(x) \]

D is a real square matrix of dimension \((2N, 2N)\). A similarity transformation, represented by \(V\), exists such that \([82], \text{ pp. 362}\),

\[ v^{-1}DV = J \]

\(J\) is the Jordan form of \(D\). Throughout this section it will be assumed that \(D\) is non-defective, meaning that every eigenvalue \(\lambda_i\) of \(D\) has a full complement of independent eigenvectors, i.e. \(D\) has \(2N\) independent eigenvectors. Then \(J\) is diagonal and the diagonal elements are the eigenvalues of \(D\). A more suggestive notation for \(J\) therefore is \(\Lambda\).

Assume now that \(D\) has \(k\) distinct eigenvalues \(\lambda_i\) with multiplicity \(m_i\). Then \(\Lambda\) can conveniently be partitioned as shown in Equation (9),

\[
\Lambda = \\
\begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \lambda_k
\end{bmatrix}
\]

Every block \(\Lambda_i\) is a diagonal matrix of dimension \((m_i, m_i)\) with \(\lambda_i\) on the diagonal. The columns of \(V\) are the corresponding right eigenvectors, as indicated by Equation (10),

\[ DV = VA \]
Since $D$ is a real matrix, both complex eigenvalues and eigenvectors occur in complex conjugate pairs. The left eigenvectors of $D$, represented by the columns in a matrix $V'$, are defined by Equation (11). The left eigenvectors will equal the right eigenvectors when $D$ is symmetric.

\[
D^t V' = V' \Lambda \\
V'^t D = \Lambda V'^t
\]

(11)

The left eigenvectors occur also in complex conjugate pairs. From Equation (10) and (11) follow that the rows of $V^{-1}$ represent a set of left hand eigenvectors,

\[
V'^t = V^{-1}
\]

(12)

Hence, when column $i$ in $V$ is the complex conjugate of column $j$ in $V$, so will row $i$ in $V^{-1}$ be the complex conjugate of row $j$ in $V^{-1}$. Finally it follows from the definition of $D$, by Equation (6a), that,

\[
V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad V_1 = V_2 \Lambda
\]

(13)

Further on it will also be useful to consider a vertical partitioning of $V$, conforming with the partitioning of $\Lambda$ as in Equation (9),
(14) \[ V = \begin{bmatrix} v_1,1 & \cdots & v_1,k \\ v_2,1 & \cdots & v_2,k \end{bmatrix} \]

Let a vector \{g\} of 2N principal coordinates, also designated modal coordinates in this context, be defined as,

\[ \{g\} = V^{-1}\{\tilde{y}\} \]
\[ \{\tilde{y}\} = V\{g\} \]

Substituting \{\tilde{y}\} in Equation (7) by \{\tilde{y}\} from Equation (15), premultiplying both sides with \(V^{-1}\) and simplifying the resulting expression using the property expressed by Equation (8), yields,

(16a) \[ \{g\} - \Lambda\{g\} = V^{-1}P\{x\} \]
(16b) \[ g_i - \lambda_i g_i = [V^{-1}P]_i\{x\} \quad i = 1 \ldots 2N \]

Hence a set of 2N non-homogeneous first order linear constant coefficient differential equations in the principal coordinates \(g_i\) is derived. The right side is a weighted summation of the forces \{x\} acting on the structure. The general solution for Equation (16a) or (16b) is,

(17a) \[ \{g\} = \int_{t_0}^{t} e^{\Lambda(t-\tau)} V^{-1}P\{x(\tau)\} \, d\tau + \{g(t_0)\}e^{\Lambda(t-t_0)} \]
(17b) \[ g_i = \int_{t_0}^{t} e^{\lambda_i(t-\tau)} [V^{-1}P]_i\{x(\tau)\} \, d\tau + g_i(t_0)e^{\lambda_i(t-t_0)} , \]
\[ i = 1 \ldots 2N \]
When the structure is at rest at $t=t_0$, and choosing $t_0=0$, then,

(18a) \[ \{g\} = \int_0^t e^{\Lambda(t-\tau)} v^{-1} P\{x(\tau)\} \, d\tau \]

(18b) \[ g_i = \int_0^t e^{\lambda_i(t-\tau)} [v^{-1} P_i] \{x(\tau)\} \, d\tau \]
\[ i = 1 \ldots 2N \]

The free decay response starting at $t_0=0$, for some set of initial conditions in the principal coordinates equal to $\{g_0\}$, equals,

(19a) \[ \{g\} = e^{\Lambda t} \{g_0\} \]

(19b) \[ g_i = g_{0i} e^{\lambda_i t} , \ i = 1 \ldots 2N \]

From Equation (2) and (15) follow that $\{g_0\}$ is related to a set of initial conditions in the original coordinate system by,

(20) \[ \{\tilde{y}_0\} = v^{-1} \{\tilde{y}_0\} \]

(21) \[ \{\tilde{y}_0\} = \begin{bmatrix} \{\dot{y}_0\} \\ \{y_0\} \end{bmatrix} \]

When the applied forces are harmonic with frequency $\omega$, as defined in Equation (22), and $j\omega$ does not coincide with an eigenvalue $\lambda_i$, then the solution for Equation (16a) and
(16b) is given by Equation (23a) and (23b),

\[(22) \quad \{x\} = \{x_0\} e^{j\omega t}\]

\[(23a) \quad \{g\} = [j\omega I - \Lambda]^{-1}V^{-1}P\{x_0\} e^{j\omega t}\]

\[(23b) \quad g_i = (j\omega - \lambda_i)^{-1}[V^{-1}P]_i\{x_0\} e^{j\omega t}, \quad i = 1 \ldots 2N\]

The latter equations are generally used to describe the resonance phenomena of structures.

Another special forcing condition exists when the only force acting on the structure is a unit impulse or Dirac impulse at coordinate \(j\), that is, \(x_j\) equals \(\delta(t)\). Denoting the response for this loading and zero initial conditions by \(p_{i,j}\), it follows from Equation (18b),

\[(24) \quad p_{i,j} = (V^{-1}P)_{i,j} \int_0^t e^{\lambda_i(t-\tau)} \delta(\tau) \, d\tau\]

But,

\[(25) \quad \int_0^t e^{\lambda_i(t-\tau)} \delta(\tau) \, d\tau = e^{\lambda_i t}\]

so that \(p_{i,j}\) is in the form of the free decay with initial conditions in the principal or modal coordinates \(p_{o_i,j}\),

\[(26a) \quad p_{o_i,j} = (V^{-1}P)_{i,j}\]

\[(26b) \quad [p_o] = V^{-1}P\]

\[(27a) \quad p_{i,j} = p_{o_i,j} e^{\lambda_i t}\]

\[(27b) \quad [p] = e^{\Lambda t}[p_o]\]
Using these expressions in Equation (18a), it follows,

\[(28) \quad \{g\} = \int_0^t [p(t-\zeta)]\{x(\zeta)\} \, d\zeta\]

Using Equation (2), (13) and (15), an expression for \{y\}, solution of the set of differential equations expressed by Equation (1), is finally obtained as,

\[(29) \quad \{y\} = \mathcal{V}_2\{g\}\]

Depending on initial conditions and forcing functions, \{g\} is given by any of the expressions, derived above.

1.3.2. Unit Impulse Response

The unit impulse response for reference location \(j\), represented by \(\{H\}_j\), is defined as the system response for unit impulse, or Dirac impulse, at location \(j\). All unit impulse responses will be represented by \(H\), a matrix of dimension \((N,N)\). Using Equation (29), with \(\{g\}\) from Equation (28), and using Equation (25) through (27),

\[(30a) \quad \{H\}_j = \mathcal{V}_2 e^{\Lambda t}\{v^{-1}p\}_j\]

\[(30b) \quad H = \mathcal{V}_2 e^{\Lambda t}v^{-1}p\]

\[(30c) \quad H = \sum_{i=1}^{2N} \{V_2\}_i e^{\lambda_i t}\{v^{-1}p\}_i\]
\[(30d)\quad H = \sum_{i=1}^{k} V_{2,i} e^{\Lambda_i t} [v^{-1} f]_i\]

In Equation (30d) a partitioning is used, conforming with the partitioning of \(\Lambda\) in Equation (9). With Equation (26b), (27b), (28) and (29), and \(H\) as defined above, it follows that for a system that is initially at rest the forced response can be written as,

\[(31)\quad \{y\} = \int_{0}^{t} H(t-\tau)\{x(\tau)\} \, d\tau\]

\(H(t)\) is also known as the Green function for the set of second order equations defined by Equation (1).

The expressions for \(H\) are independent of the selected set of independent eigenvectors. To see this, let \(S\) be a block-diagonal matrix,

\[(32)\quad S = \begin{bmatrix}
S_1 & \cdots & \cdots \\
\cdot & S_2 & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & S_k
\end{bmatrix}\]

The partitioning of \(S\) conforms with the partitioning of \(\Lambda\) as in Equation (9). For a simple eigenvalue \(\lambda_i\) the corresponding \(S_i\) is a scalar. For an eigenvalue with multiplicity \(m_i\), the corresponding \(S_i\) is an invertible matrix of dimension \((m_i, m_i)\) which maintains a set of independent eigenvectors. Let \(\tilde{V}\) be,
\[ \tilde{V} = V S^{-1} \]
\[ V = \tilde{V} S \]

Substituting \( V \) from Equation (33) in Equation (30b),

\[ H = \tilde{V}_2 S e^{\Lambda t} S^{-1} \tilde{V}^{-1} P \]

But,

\[ S e^{\Lambda t} S^{-1} = e^{\Lambda t} \]

and hence,

\[ H = \tilde{V}_2 e^{\Lambda t} \tilde{V}^{-1} P \]

which proves that the scaling of the eigenvectors does not matter.

1.3.3 Definition of Modal Parameters

Alternatively \( H \) can be expressed by Equation (37a) or (37b).

\[ (37a) \quad H = \sum_{i=1}^{2N} R_i e^{\lambda_i t} \]
\[ (37b) \quad H = \sum_{i=1}^{k} R_{ci} e^{\lambda_i t} \]

The summation in the first expression is over all eigenvalues \( \lambda_i \), all counted according to their multiplicity \( m_i \).
\( \lambda_i \) is also frequently called the pole value for mode \( i \).

Equation (38) further defines \( \alpha_i \) the modal damping factor, 
or modal attenuation, the damping ratio \( \xi_i \), the natural 
frequency \( \omega_{ni} \) and the damped natural frequency \( \omega_{di} \).

\[
\lambda_i = \alpha_i \pm \omega_{di} = -\xi_i \omega_{ni} \pm j\omega_{ni} \sqrt{1 - \frac{\xi_i^2}{2}}
\]

The matrix \( R_i \) is called the residue matrix for mode \( i \).

In Equation (37b), the summation is over all distinct eigen-
values. This notation will sometimes be convenient in latter 
developments. The matrix \( R_{ci} \) is a matrix obtained by summing 
the \( m_i \) residue matrices \( R_i \) in Equation (37a) that corre-
sponds to a mode with multiplicity \( m_i \), and will be called 
the combined residue matrix.

Comparing Equation (30c) with Equation (37a), and Equation 
(30d) with Equation (37b), it follows respectively,

\[
(39a) \quad R_i = \{V_2\}_i [V^{-1}P]_i \\
(39b) \quad R_{ci} = V_{2,i} [V^{-1}P]_i
\]

These factored forms for \( R_i \) and \( R_{ci} \) will be used extensively 
in the subsequent developments. \( P \) in this equation is real.
The complex columns in \( V \) and the complex rows in \( V^{-1} \) occur 
in complex conjugate pairs. Therefore, the complex residue 
matrices \( R_i \) and \( R_{ci} \) also occur in complex conjugate pairs.
Note also that Equation (30c) and (30d) could be derived 
under the assumption that every repeated eigenvalue has a
full complement of eigenvectors and therefore Equation (37a) and (37b), being formally equal to these equations, are also valid under this assumption.

The vectors \( \{V_2\}_i \) can be identified with free modes of vibration, or mode shapes. For \( \{V_2\}_i \) to be a mode shape, it must be demonstrated that motion can exist in any \( \{V_2\}_i \) without tending to disturb the system in any other such vector. This can be proved as follows. Using Equation (29) with \( \{g\} \) from Equation (19a), the free decay response can be written as,

\[
(40) \quad \{y\} = V_2 e^{A^t} \{g_0\}
\]

Consider the initial conditions,

\[
(41) \quad \{y_0\} = \{V_2\}_i \quad \{\dot{y}_0\} = \lambda_1 \{V_2\}_i
\]

Then \( \{g_0\} \) is determined from,

\[
(42) \quad \begin{bmatrix} V_2A \\ V_2 \end{bmatrix} \{g_0\} = \begin{bmatrix} \lambda_1 \{V_2\}_i \\ \{V_2\}_i \end{bmatrix}
\]

It follows that \( \{g_0\} \) is a zero vector except for element \( g_{0i} \) which equals 1. Therefore, the free decay response corresponding to the initial conditions specified by Equation (41) equals,
\[(43) \{y\} = \{V_2\}_i e^{\gamma_i t}\]

Hence \(\{V_2\}_i\) is a free mode of vibration. Note that for a mode with multiplicity \(m_i\), \(m_i\) independent free modes exist.

From Equation (39) it follows that the elements in \([V^{-1}P]_i\) are scale factors between the columns of the residue matrix \(R_i\). In this work, these factors will further be called modal participation factors. A modal participation factor \((V^{-1}P)_i,j\) indicates how much mode \(i\) is contributing to the response for input at generalized coordinate \(j\) of the structure. In the subsequent chapters, the symbol \(L\) will be used for the modal participation factors, rather than \(V^{-1}P\). The initial conditions in the modal coordinates \(\{g_0\}_i\), related to the initial conditions in the physical coordinates by Equation (20) and (21), that indicate how much every mode contributes to the free decay response, as expressed by Equation (40), will further on also be called modal participation factors.

The expressions derived so far for \(H\) are valid for real matrices \(M\), \(C\) and \(K\). More precise statements can be made when these matrices have specific properties.
1.3.4  Positive Definite and Symmetric Structure Parameters

By considering the quadratic forms defined by the matrices $M$, $C$ and $K$, some results can be derived that describe the location of the pole values $\lambda_i$. Any $\lambda_i$ and corresponding $\{V_2\}_i$ satisfy Equation (44),

$$ (44) \quad \lambda_i^2 M \{V_2\}_i + \lambda_i C \{V_2\}_i + K \{V_2\}_i = 0 $$

Premultiplying with $\{V_2\}_i^h$,

$$ (45a) \quad \lambda_i^2 \{V_2\}_i^h M \{V_2\}_i + \lambda_i \{V_2\}_i^h C \{V_2\}_i + \{V_2\}_i^h K \{V_2\}_i = 0 $$
$$ (45b) \quad \lambda_i^2 m_i + \lambda_i c_i + k_i = 0 $$

Hence $\lambda_i$ can be expressed as,

$$ (46) \quad 2m_i \lambda_i = -c_i \pm j\sqrt{4m_i k_i - c_i^2} $$

For the following analysis, assume $M$ to be positive definite, i.e. $m_i > 0$ for any $\{V_2\}_i$, and let $C$ and $K$ be positive semi-definite, i.e. $c_i > 0$ and $k_i > 0$ for any $\{V_2\}_i$. These assumptions hold true for most physical realisable structures.

Real values for $\lambda_i$ can only occur when $(4m_i k_i - c_i^2)$ is non-positive ($\leq 0$). From Equation (46) it then follows that $\lambda_i$ will be real and non-positive ($\leq 0$), and the system is overdamped. If $k_i = 0$, one $\lambda_i$ will be zero, and the system is neutral stable. When $(4m_i k_i - c_i^2)$ is positive, $\lambda_i$ must occur.
in complex conjugate pairs, and the system is underdamped. When \( c_i = 0 \), two complex conjugate \( \lambda_i \) exist that are pure imaginary, and the system is neutral stable. In conclusion, it follows then that when both \( C \) and \( K \) are positive definite all \( \lambda_i \) have a negative real part and the system will be stable. This is the situation observed for most physical realizable systems.

Assuming symmetric matrices for \( M \), \( C \) and \( K \), i.e. the system satisfies the Maxwell-Betti reciprocity principle, the matrices \( A \) and \( B \) as defined by Equation (2) will be symmetric. The following orthogonality relations are a classical result ([80], pp. 419-420)

\[
(47a) \quad V^t A V = \Lambda = \begin{bmatrix}
A_1 & \cdots & \cdots & \cdots \\
\vdots & A_2 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \cdots & A_k
\end{bmatrix}
\]

\[
(47b) \quad V^t B V = \tilde{\Lambda} = \begin{bmatrix}
B_1 & \cdots & \cdots & \cdots \\
\vdots & B_2 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \cdots & B_k
\end{bmatrix}
\]

\[
(47c) \quad \Lambda = \Lambda V_2^t M V_2 + V_2^t M V_2 \Lambda + V_2^t C V_2
\]

\[
(47d) \quad \tilde{\Lambda} = -\Lambda V_2^t M V_2 \Lambda + V_2^t K V_2
\]

The last two equations follow by substituting \( A \) and \( B \) from
Equation (2) and \( V \) from Equation (13) in the first two equations. The partitioning of the block diagonal matrices in these equations conforms with the partitioning of \( \Lambda \) in Equation (9). For simple eigenvalues, \( A_i \) and \( B_i \) are scalars, for an eigenvalue with multiplicity \( m_i \), \( A_i \) and \( B_i \) are symmetric matrices of dimension \( (m_i, m_i) \). The matrices \( A_i \) are necessarily invertible, since both \( A \) and \( V \) are invertible.

From Equation (6a), (8), (9) and (47a) and (47b), it follows furthermore that \( A \), \( B \) and \( \Lambda \) are related as follows,

\[
\begin{align*}
(48a) \quad & A^{-1}B = -\Lambda \\
(48b) \quad & A_i^{-1}B_i = -\Lambda_i
\end{align*}
\]

Note that the eigenvectors corresponding to an eigenvalue \( \lambda_i \) with multiplicity \( m_i \) can eventually be orthogonalized with respect to \( A \), so that \( \Lambda \) is diagonal. From Equation (48a) it follows that the corresponding \( B \) is also diagonal.

Equations (30b), (30c) and (30d) for \( H \) can now be written as,

\[
\begin{align*}
(49a) \quad & H = v_2 e^{\Lambda t \bar{A}^{-1} v_2^t} \\
(49b) \quad & H = \sum_{i=1}^{2N} \{ v_2 \}_{i} e^{\lambda_i t [\bar{A}^{-1} v_{2,i}^t]} \\
(49c) \quad & H = \sum_{i=1}^{k} v_2, i e^{\lambda_i t \bar{A}_i^{-1} v_{2,i}^t}
\end{align*}
\]

Again the partitioning in Equation (49c) conforms with the
partitioning of $\Lambda$ in Equation (9). Comparing Equation (49b) with Equation (37a) and Equation (49c) with Equation (37b), yields following expression for $R_i$ and $R_{ci}$ respectively,

\begin{align}
(50a) \quad R_i &= \{V_2\}_i^{\dagger}A^{-1}V_2^{\dagger} \\
(50b) \quad R_{ci} &= V_2^{\dagger}A^{-1}V_2^{\dagger}
\end{align}

Since every $A_i$ is symmetric, the combined residue matrices $R_{ci}$ defined by Equation (50) are symmetric, which is consistent with the fact that the system satisfies the reciprocity principle. The residue matrices $R_i$ will all be symmetric when $A$ is pure diagonal.

1.3.5 Normal Modes

It is interesting to establish the conditions for which the mode shapes, represented by $V_2$, can be real. Hereto, rewrite Equation (48a) as,

\begin{equation}
(51) \quad \Lambda \Lambda + B = 0
\end{equation}

Substituting $A$ and $B$ from Equations (47c) and (47d) in Equation (51) yields,

\begin{equation}
(52) \quad V_2^{\dagger}MV_2\Lambda^2 + V_2^{\dagger}CV_2\Lambda + V_2^{\dagger}KV_2 = 0
\end{equation}

Assuming $V_2$ to be real, it follows from equating real and
imaginary part in Equation (52) and reorganizing the resulting equations that,

(53a) \(-v_2^t M v_2^2 (\Lambda)^2_I + (\Lambda)^2_R) + v_2^t K v_2 = 0\)

(53b) \(v_2^t C v_2 + 2v_2^t M v_2 (\Lambda)_R = 0\)

\(v_2^t\) in Equation (53a) can be factored out, leaving,

(54) \(-Mv_2^2 [ (\Lambda)^2_R + (\Lambda)^2_I ] + K v_2 = 0\)

Hence \([ (\Lambda)^2_R + (\Lambda)^2_I ]\) and \(v_2\) are eigenvalues and corresponding eigenvectors of \(M^{-1}K\) in the equations of motion of the associated conservative system,

(55) \(M\{\ddot{y}\} + K\{y\} = \{x\}\)

Let \(M^{-1}K\) be non-defective. Since \(M\) is assumed positive definite, all eigenvalues of \(M^{-1}K\) are real and non-negative and can, counting their multiplicity, be represented by \(\Omega^2_n\). Let \(V_n\) be a set of corresponding linearly independent eigenvectors, so that,

(56a) \(-M V_n \Omega^2_n + K V_n = 0\)

(56b) \(V_n^t M V_n = M\)

(56c) \(V_n^t K V_n = K\)

Again, \(M\) and \(K\) are block diagonal and can be diagonal with a proper choice of the vectors corresponding to the repeated eigenvalues. Select now for \(\Lambda\) and \(v_2\),

72
(57a) \[ \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \]

(57b) \[ V_2 = [V_n, V_n] \]

Substituting \( \Lambda \) and \( V_2 \) in Equation (54) by \( \Lambda \) and \( V_2 \) from Equation (57a) and (57b), and using the property expressed by Equation (56a), it follows,

(58a) \[ (\Lambda_1)_R^2 + (\Lambda_1)_I^2 = \Omega_n^2 \]

(58b) \[ (\Lambda_2)_R^2 + (\Lambda_2)_I^2 = \Omega_n^2 \]

Substituting \( \Lambda \) and \( V_2 \) in Equation (53b) by \( \Lambda \) and \( V_2 \) from Equation (57a) and (57b), and using the property expressed by Equation (56b),

(59a) \[ V_n^t CV_n + 2M(\Lambda_1)_R = 0 \]

(59b) \[ V_n^t CV_n + 2M(\Lambda_2)_R = 0 \]

It follows from these two equations that,

(60) \[ V_n^t CV_n = \mathcal{C} \]

(61) \[ (\Lambda_1)_R = (\Lambda_2)_R = \Lambda \]

(62) \[ \mathcal{C} = -2M \Lambda \]

\( \mathcal{C} \) has a block diagonal structure similar to \( M \) and \( K \). The condition expressed by Equation (60), i.e. that the eigenvectors of \( M^{-1}K \) diagonalize \( \mathcal{C} \), is known as the condition for proportional damping. From Equations (56b) and (56c) it follows that this condition is obviously fulfilled when,
(63) \( C = aM + bK \), for \( a \) and \( b \) real

More general conditions for which the eigenvectors of \( M^{-1}K \) diagonalize \( C \) can be found in the literature ([80], pp. 391-394).

(\( \Lambda_1 \)) and (\( \Lambda_2 \)) follow further from Equation (58a), (58b) and (61), as,

\[
(64) \quad (\Lambda_1)^2 = (\Lambda_2)^2 = \Omega_d^2
\]
\[
(65) \quad \Omega_d = \sqrt{\Omega_n^2 - \Lambda^2}
\]

To have all elements in \( \Lambda \) in complex conjugate pairs, select,

\[
(66a) \quad \Lambda_1 = A + j\Omega_d
\]
\[
(66b) \quad \Lambda_2 = A - j\Omega_d
\]

Substituting \( \Lambda \) from Equation (57a) and \( V_2 \) from Equation (57b) in Equation (47c), and using the properties expressed by Equation (60), (62), (66a) and (66b), a simplified expression for \( A \) is derived,

\[
A = j2\begin{bmatrix}
\Omega_d M & 0 \\
0 & -\Omega_d M
\end{bmatrix}
\]

(67)

\[
A = j2(\Lambda)^\top[N \otimes M]
\]

Hence \( A \) is pure imaginary. Since \( V_2 \) is real, it follows that all residue matrices defined by Equation (50) are pure
imaginary.

1.3.6 General Definition of Modal Mass

For the general case, the matrix of impulse responses expressed in terms of modal parameters equals (see Equation (30b), but using $L$ for $V^{-1}P$),

\[(68a) \quad H = V_2 e^{\Lambda t} L\]

\[(68b) \quad H = \sum_{i=1}^{k} V_{2,i} e^{\Lambda_i t} L_i\]

Comparing these expressions with the ones obtained for symmetric matrices, Equation (49a) and (49c), it follows,

\[(69a) \quad L = A^{-1} V_2^T\]

\[(69b) \quad L_i = A_i^{-1} V_{2,i}^T, \quad i = 1 \ldots k\]

The above equations indicate that, for the case of symmetric matrices, the modal participation factors are not independent of the mode shape coefficients. For every mode with pole value $\lambda_i$ and multiplicity $m_i$, the corresponding $L_i$ is related to $V_{2,i}$ by the inverse of a symmetric matrix $A_i$, that can be diagonal for a proper choice of the mode shapes. Historically, modal mass is been defined as,
(70a) \[ M_m = \frac{A}{2j(\Lambda)_I A_i} \]

(70b) \[ M_{mi} = \frac{A_i}{2j(\Lambda_i)_I} \]

The term modal mass is related to the fact that for a proportional damped system, using \( A \) from Equation (67), above equations simplify to,

(71) \[ M_m = [N \otimes N] \]

For a non-repeated mode, \( M_{mi} \) represents a unique number, for a repeated mode with multiplicity \( m_i \), \( M_{mi} \) represents an invertible \((m_i, m_i)\) matrix. In either case, the numerical value of \( M_{mi} \) is determined by the scaling of the corresponding mode shapes \( V_{2,i} \).
CHAPTER 2  THEORETICAL DEVELOPMENT

This chapter deals with a detailed discussion of the theoretical development of the frequency and time domain direct parameter model identification methods, applicable for multiple input modal analysis, that have been examined in this dissertation.

In the first section, a general model is derived for the equations of motion of a viscous damped mechanical structure. In the continuous time domain, this model is in the form of a higher order linear differential equation for a multivariate response vector, with a non-homogeneous part that depends on a multivariate force input vector and its derivatives. In the Laplace domain, this differential equation transforms into a rational matrix polynomial that represents a direct parameter model, identifiable from frequency domain measurement data. In the discrete time domain, the differential equation can be approximated by a finite difference equation, that constitutes a direct parameter
model, identifiable from uniformly sampled time domain data. In the z domain, this finite difference equations transforms into a rational matrix polynomial. Procedures will be discussed to calculate the modal parameters from the direct parameter model either in the Laplace or z domain, including the situation of repeated modes that have a full complement of independent mode shapes.

Procedures for estimating the parameters in a direct parameter model from frequency domain data have chronologically been developed first, building and expanding on earlier work [44, 45, 48, 76-78], and are discussed in the second section. It will be shown that the numerical conditioning of these estimation procedures becomes critical for many situations, therefore requiring special precaution and limiting the applicability for a general purpose implementation.

Section three discusses estimation procedures for time domain direct parameter model identification. These procedures turn out to be much better conditioned than the corresponding frequency domain procedures. For a number of cases of practical interest, it is also possible to apply special solution methods that reduce considerably the overall execution time. For some kinds of data, such as impulse response functions and free decay response data, the
procedure can also be modified to estimate in a first phase only the parameters in the homogeneous part of the finite difference equation. This provides all information for calculating pole values and mode shapes. In a second phase then, the non-homogeneous part of the finite difference equation, or modal participation factors, are estimated. This two phase procedure is then intrinsically related to such time domain modal model identification methods as the Least Squares Complex Exponential [39,40], the Polyreference method [41-43] and the Ibrahim Time Domain method [30-34].

In the fourth section, a discussion is given of some aspects of modal parameter calculation, validation and post-processing, with emphasis on multiple input modal analysis.

Finally, section five concludes this chapter with an overview of the various direct parameter model identification methods, examined in this dissertation.
2.1 Theoretical Derivation of Frequency and Time Domain Direct Parameter Models and their Relation with the Modal Parameters of a Mechanical Structure

The purpose of this section is to present a theoretical derivation and justification of the estimation models that have been used for the direct parameter model identification methods, examined in this dissertation.

First a general differential equation form is derived for the response at $N_0$ locations of a structure with $N$ effective degrees of freedom and force application at $N_1$ locations.

Frequency domain models, i.e. parametric relations between the Fourier transformed measurement data, are derived directly from the Laplace transform image of the differential equation form. Time domain models, i.e. parametric relations between the uniformly sampled measurement data, are derived using both the general differential equation form and the canonical response expressions derived in Section 1.3.

Not surprisingly the frequency and time domain models will be seen to be very analogous. The parameters in either of
the models are related to the modal parameters, discussed in Section 1.3.3.

The direct parameter model identification methods have been applied primarily to obtain global estimates of the modal parameters. The characteristics and usage of the modal parameters obtained from the direct parameter models will therefore be discussed in detail.

2.1.1 General Differential Equation Form for the Equations of Motion of a Mechanical Structure

Let a linear, lumped parameter, viscous damped, mechanical structure be excited at \( N_1 \) input locations, and let the forced response be measured at \( N_o \) output or response locations. Let the response be composed of \( N \) independent components, or degrees of freedom.

Initially, it will be assumed that there are as many response locations as there are degrees of freedom and that the \( N_o \) response locations constitute a set of generalized coordinates. That is, let the responses at the \( N_o \) response locations represent independent variables. It is also assumed that the \( N_1 \) input locations are a subset of the \( N_o \) response locations. Let \( \{x\} \) be a vector of \( N_1 \) elements,
representing the excitation at the \( N_i \) input locations, and let \( \{y\} \) be a vector of \( N_o \) elements with the response at the \( N_o \) response locations. The forced response of this mechanical system is described by a set of \( N_o \) simultaneous, linear, constant coefficient, second order differential equations, with as dependent variables the response \( \{y\} \) at the \( N_o \) response locations (such an equation system will be designated a linear, second order matrix differential equation of dimension \((N_o, N_o)\) with \( \{y\} \) as dependent variable),

\[
(1) \quad M\ddot{y} + C\dot{y} + K\{y\} = E\{x\}
\]

In this equation, \( M, C \) and \( K \) are matrices of dimension \((N_o, N_o)\) and stand for respectively the mass, stiffness and damping matrices, lumped at the \( N_o \) response locations. The matrix \( E \) is of dimension \((N_o, N_i)\), with element \( e_{i,j} \) equal to 1 when input location \( j \) corresponds to response location \( i \), and equal to 0 otherwise. As was argued in Section 1.3.1, the matrix \( M \) in Equation (1) will be invertible since it is assumed that this equation represents the response of a \( N_o \) degree of freedom system. Therefore, Equation (1) can be written in a more condensed form,

\[
(2) \quad \dot{\{\ddot{y}\}} + M^{-1}C\{\dot{y}\} + M^{-1}K\{y\} = M^{-1}E\{x\}
\]

or using a simplified notation,

\[
(3) \quad \ddot{\{\ddot{y}\}} + C'\{\dot{y}\} + K'\{y\} = Q\{x\}
\]
$M^{-1}E$, or $Q$, is a matrix of dimension $(N_o, N_i)$, that represents the $N_i$ columns in the matrix $M^{-1}$ corresponding to the $N_i$ input locations.

For some applications, it may be that the response of a system is only important at a small number of locations. For example, the response of a component may only be required at the connection points. So let the response be available at $N_o$ response locations, $N_o$ smaller than $N$, and,

$$N = pN_o,$$

$p$ a positive non-zero integer.

The low order complete model, derived for the case with $N_o$ equal to $N$, can be modified to a low order complete model applicable for this situation. Suppose the response to be measured at $(N-N_o)$ additional, fictitious, response locations on the structure, that together with the $N_o$ response locations form a set of $N$ generalized coordinates. Let the forced response at all $N$ response locations be represented by a vector $\{\tilde{y}\}$. With the vector $\{x\}$, the force input at $N_i$ locations that are a subset of all $N$ response locations and not necessarily of the $N_o$ response locations, the forced response $\{\tilde{y}\}$ is described by a linear second order matrix differential equation of dimension $(N, N)$ like Equation (3),

$$\ddot{\{\tilde{y}\}} + C'\{\tilde{y}\} + k'\{\tilde{y}\} = Q\{x\}$$

The response $\{\tilde{y}\}$ can be partitioned equally in $p$ parts, with
the first partition \( \{ \tilde{y}_1 \} \) representing the response at the \( N_0 \) actual response locations,

\[
\{ \tilde{y} \} = \begin{cases} 
\tilde{y}_1 \\
\vdots \\
\tilde{y}_p 
\end{cases}
\]

(6) \( \{ \tilde{y} \} = \begin{cases} 
\tilde{y}_1 \\
\vdots \\
\tilde{y}_p 
\end{cases} \)

Accordingly, Equation (5) can be partitioned, yielding a set of \( p \) simultaneous, linear, second order matrix differential equations of dimension \((N_0, N_0)\) with \( p \) dependent variables, \( \{ \tilde{y}_1 \}, \ldots, \{ \tilde{y}_p \} \),

(7) \[
\begin{align*}
\dddot{\tilde{y}}_i & + C_{i,j} \dot{\tilde{y}}_j & + K_{i,j} \tilde{y}_j & = Q_i(x) , \quad i = 1 \ldots p \\
\end{align*}
\]

The matrices \( C_{i,j} \) and \( K_{i,j} \) are of dimension \((N_0, N_0)\) and the matrices \( Q_i \) are of dimension \((N_0, 1)\). The vectors \( \{ \tilde{y}_2 \}, \ldots, \{ \tilde{y}_p \} \) represent the response at the additional fictitious \((N-N_0)\) response locations. It will be proved that a linear, matrix differential equation of order \( 2p \) and dimension \((N_0, N_0)\) with \( p \) dependent variable \( \{ \tilde{y}_1 \} \) can be derived, assuming that \( \{ \tilde{y} \} \) has a derivative of order \( 2p \). Let both sides of Equation (7) be differentiated up to \( 2(p-1) \) times, i.e.,

(8) \[
\begin{align*}
I(d/dt)^{k+2} \tilde{y}_i & + C_{i,j} (d/dt)^{k+1} \dot{\tilde{y}}_j & + K_{i,j} (d/dt)^k \tilde{y}_j & = Q_i (d/dt)^k x \\
\end{align*}
\] 

\[
k = 0 \ldots 2(p-1) \\
i = 1 \ldots p 
\]
All equations like Equation (8), for \( k = 0 \ldots 2(p-1) \) and \( i = 1 \ldots p \) form a set of \((2p^2-p)\) simultaneous linear matrix differential equations of order \(2p\) and dimension \((N_0,N_0)\). The response vectors \(\tilde{y}_2, \ldots, \tilde{y}_p\) and their derivatives up to order \(2p\) are unknown, in total \((2p+1)(p-1)\) unknown vectors of \(N_0\) elements. Hence there is exactly one matrix differential equation more in the set of equations specified above than there are unknown vectors, so that the unknowns can be eliminated to obtain one linear matrix differential equation of order \(2p\) and dimension \((N_0,N_0)\) with \(\tilde{y}_1\) as dependent variable,

\[
(9) \quad I(d/dt)^{2p}\tilde{y}_1 + \lambda_1(d/dt)^{2p-1}\tilde{y}_1 + \ldots + \lambda_{2p}\tilde{y}_1 = B_0(d/dt)^{2p-2}x + \ldots + B_{2p-2}x
\]

The existence of the derivative of order \(2p\) of \(\tilde{y}\) and the subsequent derivation of Equation (9) by elimination, is justified by the assumption that \(\tilde{y}_1\) represents the response at \(N_0\) response locations of a mechanical system with \(N\) degrees of freedom, and \(N = pN_0\). This assumption implies that the forced response is described by a linear matrix differential equation that has \(2N\) characteristic solutions. The linear matrix differential equation that is derived, Equation (9), has as a characteristic equation,

\[
(10) \quad |s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p}| = 0
\]
Since $I, A_1, \ldots, A_{2p}$ are all matrices of dimension $(N_0, N_0)$, the characteristic equation defined by Equation (10) will be a polynomial equation of order $2N$ in $s$. Hence Equation (9) has $2N$ characteristic solutions, as required by the specified assumption.

The characteristic solutions, restricted to the $N_0$ response locations of interest, of the matrix differential equation defined by Equation (5) are also equal to those of the matrix differential equation defined by Equation (9). A characteristic solution of Equation (5) is a solution of the homogeneous equation,

\[(11) \quad \ddot{\{y\}} + C'\{\dot{y}\} + K'\{y\} = 0\]

Let $\lambda$ and $\{V\}$ be an eigenvalue and right eigenvector of the characteristic polynomial of Equation (11), i.e.,

\[(12) \quad [\lambda^2I + \lambda C' + K']\{V\} = 0\]

Assuming that every repeated eigenvalue has a full complement of independent eigenvectors, it then follows from Equation (12) that a solution for Equation (11) is of the general form,

\[(13) \quad \{\ddot{y}\} = \{V\}e^{\lambda t}\]

Let $\{V\}$ be partitioned equally in $p$ parts, conforming to the partitioning of $\{\ddot{y}\}$ defined by Equation (6),
\[
\{V\} = \begin{cases}
\{V_1\} \\
\vdots \\
\{V_p\}
\end{cases}
\]

The solution of Equation (11), restricted to the \(N_0\) response locations of interest, \(\tilde{y}_1\), is also a solution of the homogeneous equation,

\[
I(d/dt)^2p\{\tilde{y}_1\} + A_1(d/dt)^2p-1\{\tilde{y}_1\} + \ldots + A_{2p}\{\tilde{y}_1\} = 0
\]

Substituting \(\tilde{y}\) from Equation (6) and \(\{V\}\) from Equation (14) into Equation (13), it follows that the solution for Equation (15) is of the form,

\[
\{\tilde{y}_1\} = \{V_1\}e^{\lambda t}
\]

Finally, after substituting \(\tilde{y}_1\) from Equation (16) into Equation (15), it is derived that \(\lambda\) and \(\{V_1\}\) are eigenvalues and corresponding eigenvectors of the characteristic polynomial of Equation (9),

\[
[\lambda^{2p}I + \lambda^{2p-1}A_1 + \ldots + A_{2p}]\{V_1\} = 0
\]

Substituting \(\tilde{y}_1\) by \(y\), Equation (9) can be expressed with simplified notation as,

\[
I(d/dt)^2p\{y\} + A_1(d/dt)^2p-1\{y\} + \ldots + A_{2p}\{y\} = B_0(d/dt)^2p-2\{x\} + \ldots + B_{2p-2}\{x\}
\]

Equation (18) is a general differential equation form for
the equations of motion of a mechanical structure with \( N \) degrees of freedom, force input at \( N_1 \) locations and response signals at \( N_0 \) locations, \( N_0 \) being related to \( N \) by Equation (4).

As a special case, with \( p \) equal to 1 and with \( B_{-1} \) and \( B_{-2} \) both equal to 0, Equation (18) becomes,

\[
(19) \quad I\ddot{y} + A_1\dot{y} + A_2y = B_0x
\]

This equation is formally equal to Equation (3). It is however worth noting that the derivation of Equation (19) as a special case of Equation (18) does not require \( \{x\} \) to represent the force input at \( N_1 \) input locations that are a subset of the \( N_0 \) response locations, as was assumed for the derivation of Equation (3) from Equation (1). This assumption is therefore unnecessarily restrictive when the equations of motion are expressed in the form of Equation (3) or Equation (19).

Finally, let the number of response locations \( N_0 \) be larger than the number of degrees of freedom, \( N \). Then all response locations can no longer define a set of generalized coordinates. That is, the responses at the \( N_0 \) response locations do not represent \( N_0 \) independent variables. It can still be assumed that \( \{y\} \) represents the response of a \( N_0 \) degree of freedom system described by Equation (19), with however, out
of all \( N_0 \) modes, only \( N \) modes that contribute to the response. Such a situation might occur when the force input is at locations that are all nodes for the remainder \( (N_0 - N) \) modes. Then the model defined by Equation (19) becomes a large order incomplete model. From an identification point of view, this implies that the matrices \( A_1, A_2 \) and \( B_0 \) in Equation (19) are not uniquely defined by the data. As was discussed in Section 1.2.1, this is the underlying idea of the ISSPA method: matrices \( A_1 \) and \( A_2 \) can be estimated using a minimum norm pseudo-inverse procedure.

Assume that \( (N_0 - N) \) holonomic constraint relations exist within the elements of the response vector \( \{y\} \) at the \( N_0 \) response locations. Then, with \( \{y'\} \) representing the response at \( N \) response locations that constitute a set of generalized coordinates, the following linear dependence relation can be written,

\[ (20) \quad \{y\} = H\{y'\} \]

Herein, \( H \) is a real matrix of dimension \( (N_0, N) \). The response vector \( \{y'\} \) is related to the force input vector \( \{x\} \) by a model, like defined by Equation (19),

\[ (21) \quad I\{y'\} + A_1\{y'\} + A_2\{y'\} = B_0\{x\} \]

This equation describes again a low order complete model. \( A_1 \) and \( A_2 \) are matrices of dimension \( (N, N) \), and \( B_0 \) of dimension


(N, N_1). Equation (20) and (21), with \{y^t\} the impulse responses at N response locations that are a subset of the N_0 response locations, constitute the fundamental equations for the SFD technique, discussed in Section 1.2.2.

Rather than letting \{y^t\} being the response at a subset of the N_0 response locations, and following suggestions by Vold [83], principal component analysis of vector valued variates can be used to determine both a suitable transformation matrix H and the corresponding \{y^t\} in Equation (20). This approach is pursued in this work to handle the situation where N is smaller than N_0.

The direct parameter model identification methods basically will represent estimation schemes for the matrix parameters in the matrix differential equation defined by Equation (18) from \{x\} and \{y\}, or when N_0 is larger than N from \{x\} and \{y^t\}, defined by Equation (20). The Laplace transform image of this matrix differential equation is a rational matrix polynomial in the Laplace variable s and is useful for estimation purposes using the Fourier transformed data. The modal parameters are calculated next from the partial fraction expansion of this rational matrix polynomial.

On the other hand, when the response and force input are uniformly sampled with increment Δt, it will be proved that the sampled time series are related by a finite difference

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equation that is in the form of an autoregressive moving-average equation with autoregressive part of order 2p and moving-average part of order \((2p-1)\). The matrix coefficients in the finite difference equation differ from the matrix coefficients in the differential equation. They are related, but such interrelations will not be pursued in this work. For the examined application areas, which are the extraction of modal parameters and the dynamic response prediction, the matrix coefficients in the finite difference relation can equally well be used. The Z-transform image of this finite difference equation becomes a rational matrix polynomial in the Z-transform variable \(z\), and the modal parameters are again identifiable from the partial fraction expansion.

Both direct parameter models will be developed in the next two subsections. The actual estimation schemes will be set up in Sections 2.2 and 2.3.

2.1.2 Frequency Domain Direct Parameter Models

In this section models will be derived from the matrix differential equation defined by Equation (18) for estimation purposes from Fourier transformed measurement data, or frequency domain data. Hereto, let \(\{X\}\) and \(\{Y\}\) represent the Laplace transform of \(\{x\}\) and \(\{y\}\) respectively,
\[ \{X\} = \text{La}\{\{x\}\} \]
\[ \{Y\} = \text{La}\{\{y\}\} \]

Assuming the initial conditions on the response \( \{y\} \) to be \( \{y_0^{(0)}\}, \{y_0^{(1)}\}, \ldots, \{y_0^{(k-1)}\} \), the Laplace transform of \((d/dt)^k\{y\}\) is given by (see for example [84], Chapter 2),

\[ \text{La}( (d/dt)^k\{y\} ) = s^k\{Y\} - \sum_{i=1}^{k} s^{i-1}\{y_0^{(k-i)}\} \]

Using this property and assuming that \( \{x_0^{(0)}\}, \{x_0^{(1)}\}, \ldots, \{x_0^{(2N-3)}\} \) are all zero, the Laplace transform image of the matrix differential equation defined by Equation (18) is given by Equation (24). For convenience in further developments, the matrix coefficients \( B_0, \ldots, B_{2p-2} \) are renumbered to \( B_2, \ldots, B_{2p} \).

\[ \text{[s}^2\text{PI} + s^{2p-1}A_1 + \ldots + A_{2p}]\{Y\} \]
\[ = [s^{2p-2}B_2 + \ldots + B_{2p}]\{X\} \]
\[ + s^{2p-1}\{C_1\} + \ldots + \{C_{2p}\} \]

with,
\[ \{C_1\} = \{y_0^{(0)}\} \]
\[ \{C_k\} = \{y_0^{(k-1)}\} + A_1\{y_0^{(k-2)}\} + \ldots + A_k\{y_0^{(0)}\} \]
\[ k = 2 \ldots 2p \]

Data that are produced in experiments designed for vibration analysis of mechanical structures can reasonably be assumed to belong to either of following two classes:
1. Transient response data, or wavelets, with special forcing conditions and zero initial conditions, or no force inputs and specified initial conditions.

2. Stationary forced response data. This may be expected to occur when stationary force inputs are applied and after all transient effects from initial disturbances have died out.

In either case, the force input can be stochastic or deterministic, or a combination of both. Examples are uncorrelated random inputs, chirp signals (fast sine sweep) and burst random (uncorrelated random input during the initial part of the sample period and no input during the remainder part) [18,22].

To distinguish between both types of data, it is useful to represent the response \( \{y\} \) as a superposition of the particular response \( \{y_p\} \) and the homogeneous response \( \{y_h\} \),

(25) \[ \{y\} = \{y_p\} + \{y_h\} \]

where,

(26) \[ I(d/dt)^{2p}\{y_p\} + A_1(d/dt)^{2p-1}\{y_p\} + \ldots + A_{2p}\{y_p\} \]

\[ = B_0(d/dt)^{2p-2}\{x\} + \ldots + B_{2p-2}\{x\} \]

\[ \{y_h^{(k)}\} = 0 \ , \ k = 0 \ldots 2p-1 \]

(27) \[ I(d/dt)^{2p}\{y_h\} + A_1(d/dt)^{2p-1}\{y_h\} + \ldots + A_{2p}\{y_h\} = 0 \]
\{y^{(k)}_{h_0}\} = \{y^{(k)}_0\}, \ k = 0 \ldots 2p-1

The Laplace transform images of Equation (26) and (27) become (after renumbering the coefficients \(B_0, \ldots, B_{2p-2}\) as done for Equation (24)),

\begin{align*}
(28) \quad & [s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p}]\{Y_p\} \\
& = [s^{2p-2}B_2 + \ldots + B_{2p}]\{X\}
\end{align*}

\begin{align*}
(29) \quad & [s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p}]\{Y_{h}\} \\
& = s^{2p-1}\{C_1\} + \ldots + \{C_{2p}\}
\end{align*}

The coefficients \(\{C_i\}\) are defined as for Equation (24).

Adding Equation (28) and (29) side by side and using the linearity of the Laplace transform, results again in Equation (24). Equation (24) then simplifies to Equation (28) when \(\{y\}\) is transient response data with zero initial conditions or stationary response data, and to Equation (29) when \(\{y\}\) represents transient response data with specified initial conditions and no force inputs. In the following developments, the subscript \(p\) in Equation (28) and \(h\) in Equation (29) will be omitted.

Sometimes, it may be convenient to distinguish different sets of initial conditions and to rewrite Equation (29) more explicitly as,
(30) \[ s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p} \{Y\}_i \]
\[ = s^{2p-1}\{C_1\}_i + \ldots + \{C_{2p}\}_i \]

The subscript \(i\) counts different sets of initial conditions. Assuming \(N_i\) sets of initial conditions and with \(Y\) a matrix of dimension \((N_0, N_i)\) representing the Laplace transform of the response at \(N_0\) locations for the different sets, all equations like Equation (30) can be written as one matrix equation,

(31) \[ s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p}Y = s^{2p-1}C_1 + \ldots + C_{2p} \]

The coefficients \(C_1, \ldots, C_{2p}\) are all matrices of dimension \((N_0, N_i)\).

Let \(H(t)\) be a matrix of dimension \((N_0, N_i)\) with the impulse response functions between the \(N_0\) response locations and \(N_i\) reference or input locations. By definition, \(h_{i,j}(t)\) is the response at response location \(i\) for a Dirac impulse \(\delta(t)\) applied at location \(j\). Equation (32) and (33) define the Laplace transform of \(H(t)\) and \(\delta(t)\) respectively.

(32) \[ H(s) = \text{La}(H(t)) \]
(33) \[ \text{La}(\delta(t)) = 1 \]

The impulse response being a transient response, it follows using Equation (28) that,
\[(34) \quad [s^{2p}i + s^{2p-1}A_1 + \ldots + A_{2p}]\{H(s)\}_i = s^{2p-2}\{B_2\}_i + \ldots + \{B_{2p}\}_i, \quad i = 1 \ldots N_1\]

or, \[(35) \quad [s^{2p}i + s^{2p-1}A_1 + \ldots + A_{2p}]H(s) = s^{2p-2}B_2 + \ldots + B_{2p}\]

Both equations are formally equal to Equations (30) and (31), with \(C_1\) equal to 0. This is not surprising since the impulse response can alternatively be represented as the transient response with no force input and specified initial conditions, among which necessarily \(\{y_0^{(0)}\}\) equal to 0 ([85], pp. 53-55).

Equation (28) can be used to estimate the parameters \(A_1, \ldots, A_{2p}\) and \(B_2, \ldots, B_{2p}\) from Fourier transformed measurement data. Let \(\{X_k\}\) and \(\{Y_k\}\) represent the values of the Fourier transform of \(\{x\}\) and \(\{y\}\) at frequency \(\omega_k\), \(k = 1 \ldots N_f\). It will not be required in the development that the frequencies \(\omega_k\) are equally spaced. Since the value of the Fourier transform equals the value of the Laplace transform for \(s\) equal to \(j\omega_k\), \(\{X\}_k\) and \(\{Y\}_k\) can be used in Equation (28) to yield.

\[(36) \quad [s^{2p}k + s^{2p-1}A_1 + \ldots + A_{2p}]\{Y_k\} = [s^{2p-2}B_2 + \ldots + B_{2p}]\{X_k\}\]

\(s_k = j\omega_k, \quad k = 1 \ldots N_f\)

The \(N_f\) frequencies \(\omega_k\) cover only a finite frequency range.
Therefore, the effectiveness of the estimation model defined by Equation (36) can be increased when a modified response \( \{Y'_k\} \), with the residual effects from modes outside this finite frequency range removed, is used instead of \( \{Y_k\} \). This modified response is defined by,

\[
(37) \quad \{Y'_k\} = \{Y_k\} - [U - s_k^{-2}L]\{X_k\} , \quad s_k = j\omega_k
\]

Herein, \( L \) and \( U \) are matrices of dimension \((N_0, N_1)\) with the lower and upper residuals. The lower residuals account for the contribution to the response by modes at frequencies lower than any \( \omega_k \), the upper residuals account for the contribution by modes at frequencies higher than any \( \omega_k \) [13]. Fitting a rational matrix polynomial to \( \{Y'_k\} \), using the model defined by Equation (36), will require less terms, that is a model with smaller \( p \) suffices, than the rational matrix polynomial fitted to \( \{Y_k\} \), in which more terms are required to describe the contribution of modes outside the range covered by the \( N_f \) frequencies \( \omega_k \).

Using a proper data processing sequence, to be discussed in Section 2.5, it is possible to eliminate the lower residuals \( L \) implicitly. By eliminating \( L \) from the model, the estimation scheme will also be better conditioned. Substituting \( \{Y_i\} \) in Equation (36) by \( \{Y'_i\} \) from Equation (37) with \( L \) equal to 0, and after reorganizing terms, the following equation is derived,
(38) \[ s_k^{2p} I + s_k^{2p-1} A_1 + \ldots + A_{2p} \{ Y_k \} \]
\[ = [s_k^{2p} B_0' + s_k^{2p-1} B_1' + \ldots + B_{2p}'] \{ X_k \} \]
\[ s_k = j\omega_k, \quad k = 1 \ldots N_f \]

with,
\[ B_0' = U \]
\[ B_i' = B_i + A_i U, \quad i = 1 \ldots 2p \]

The model defined by Equation (38) can be used to estimate
\[ A_1, \ldots, A_{2p} \text{ and } B_0', \ldots, B_{2p}' \]. Using Equations (39), estimates for \( U, B_1, \ldots, B_{2p} \) are obtained, i.e.,

(40) \[ [s_k^{2p} I + s_k^{2p-1} A_1 + \ldots + A_{2p}] \{ Y_k \} - U \{ X_k \} \]
\[ = [s_k^{2p} B_1 + \ldots + B_{2p}] \{ X_k \} \]
\[ s_k = j\omega_k, \quad k = 1 \ldots N_f \]

or,

(41) \[ [s^{2p} I + s^{2p-1} A_1 + \ldots + A_{2p}] \{ Y \} - U \{ X \} \]
\[ = [s^{2p} B_1 + \ldots + B_{2p}] \{ X \} \]

Theoretically, \( B_1 \) in these equations should come out equal
to 0.

Similarly, with \( Y_k \) the values of the Fourier transform at
frequency \( \omega_k \) of the response at \( N_0 \) locations for \( N_1 \) sets of
initial conditions, Equation (31) becomes,

(42) \[ [s_k^{2p} I + s_k^{2p-1} A_1 + \ldots + A_{2p}] Y_k \]
\[ = s_k^{2p-1} C_1 + \ldots + C_{2p-1} + s_k = j\omega_k \]
\[ k = 1, \ldots, N_f \]
Analogous to Equation (38), (39), (40) and (41) then follow,

\[(43) \quad [s_k^{2p}I + s_k^{2p-1}A_1 + \ldots + A_{2p}]Y_k\]
\[= s_k^{2p}C_0' + s_k^{2p-1}C_1' + \ldots + C_{2p}'\]
\[s_k = j\omega_k, \quad k = 1 \ldots N_f\]

\[C_0' = U\]
\[C_i' = C_i + A_1U, \quad i = 1, \ldots, 2p\]

\[(44)\]

\[(45) \quad [s_k^{2p}I + s_k^{2p-1}A_1 + \ldots + A_{2p}][Y_k - U]\]
\[= s_k^{2p-1}C_1 + \ldots + C_{2p}, \quad s_k = j\omega_k\]
\[k = 1 \ldots N_f\]

\[(46) \quad [s^{2p}I + s^{2p-1}A_1 + \ldots + A_{2p}][Y - U]\]
\[= s^{2p-1}C_1 + \ldots + C_{2p}\]

The Fourier transformed impulse response functions, generally designated the frequency response functions, will be represented by \(H(\omega)\). With \(H(\omega_k)\), abbreviated to \(H_k\), representing the value of the frequency response functions at frequency \(\omega_k\), Equation (35) becomes,

\[(47) \quad [s_k^{2p}I + s_k^{2p-1}A_1 + \ldots + A_{2p}]H_k\]
\[= s_k^{2p-2}B_2 + \ldots + B_{2p}, \quad s_k = j\omega_k\]
\[k = 1 \ldots N_f\]

Using \(I\) and \(H_k\) instead of \(\{X_k\}\) and \(\{Y_k\}\), Equation (38) can be written as,
(48) \[ \begin{align*}
(\mathbf{s}_{k}^{2\mathbf{P}} + \mathbf{s}_{k}^{2\mathbf{P}-1}\mathbf{A}_{1} + \cdots + \mathbf{A}_{2\mathbf{P}})\mathbf{H}_{k} & = \mathbf{s}_{k}^{2\mathbf{P}}\mathbf{B}_{0}^{\prime} + \cdots + \mathbf{B}_{2\mathbf{P}}^{\prime}, \quad \mathbf{s}_{k} = j\omega_{k} \\
& \quad k = 1 \ldots \mathbf{N}_{f}
\end{align*} \]

Using Equation (39), equations similar to Equations (40) and (41) follow,

(49) \[ \begin{align*}
(\mathbf{s}_{k}^{2\mathbf{P}} + \mathbf{s}_{k}^{2\mathbf{P}-1}\mathbf{A}_{1} + \cdots + \mathbf{A}_{2\mathbf{P}})(\mathbf{H}_{k} - \mathbf{U}) & = \mathbf{s}_{k}^{2\mathbf{P}-1}\mathbf{B}_{1} + \cdots + \mathbf{B}_{2\mathbf{P}}, \quad \mathbf{s}_{k} = j\omega_{k} \\
& \quad k = 1 \ldots \mathbf{N}_{f}
\end{align*} \]

or,

(50) \[ \begin{align*}
(\mathbf{s}^{2\mathbf{P}} + \mathbf{s}^{2\mathbf{P}-1}\mathbf{A}_{1} + \cdots + \mathbf{A}_{2\mathbf{P}})(\mathbf{H}(\mathbf{s}) - \mathbf{U}) & = \mathbf{s}^{2\mathbf{P}-1}\mathbf{B}_{1} + \cdots + \mathbf{B}_{2\mathbf{P}}
\end{align*} \]

(51) \[ \mathbf{H}(\mathbf{s}) = \frac{\mathbf{s}^{2\mathbf{P}-1}\mathbf{B}_{1} + \cdots + \mathbf{B}_{2\mathbf{P}}}{\mathbf{s}^{2\mathbf{P}} + \cdots + \mathbf{A}_{2\mathbf{P}}} + \mathbf{U} \]

The latter equation simply expresses \( \mathbf{H}(\mathbf{s}) \) as a rational matrix polynomial; evaluated for \( \mathbf{s} \) equal to \( j\omega_{k} \), it represents the value of the frequency response function matrix at frequency \( \omega_{k} \).

When \( \mathbf{H}(\mathbf{s}) \) represents a single frequency response function, then Equation (51) becomes a rational scalar polynomial and equals the model used by Richardson and Formenti [27]. The order \( \mathbf{p} \) in the polynomials of this model equals the number of effective modes, \( \mathbf{N} \).

In the present formulation, where \( \mathbf{H}(\mathbf{s}) \) represents a matrix
of dimension \((N_o, N_i)\) with the frequency response functions between \(N_o\) response locations and \(N_i\) reference locations, the matrices in the denominator polynomial of Equation (51) are of dimension \((N_o, N_o)\) and \(p\) is as defined by Equation (4). Alternatively, from Maxwell-Betti reciprocity, the role of response and reference locations can be interchanged, and \(H(s)^t\) can be expressed by a model in the form of Equation (51). The matrices in the denominator polynomial of such model are of dimension \((N_i, N_i)\), and the order \(p\) is selected so that \(pN_i\) equals the number of effective modes, \(N\). In general, the number of response locations is a lot larger than the number of reference locations, requiring \(p\) to be larger for the latter approach. In Section 2.2 it will be shown that this has a detrimental effect on the numerical conditioning of the estimation process.

The actual estimation schemes for the parameters in the models defined by Equation (38), (43) and (48) and the conditions for which the parameters are uniquely defined by the data will be discussed in Section 2.2.

The remainder of this section is a derivation of a partial fraction expansion for the rational matrix polynomial defined by Equation (51), or implicitly by Equation (41) and (50). This partial fraction expansion will be derived under the assumption that every repeated eigenvalue of the
polynomial,

\[ (52) \quad [s^{2p}I + A_1s^{2p-1} + \ldots + A_{2p}] \]

has a full complement of independent eigenvectors. Under this assumption, modal parameters as defined in Section 1.3.3, are identifiable with the parameters in this partial fraction expansion.

In Section 1.3.2 a partial fraction expansion for \( H(t) \) was defined by Equations (1.3.30). Restricted to \( N_o \) response locations of interest, for \( N_i \) reference locations and assuming \( N \) modes, this partial fraction expansion can be written as,

\[ (53) \quad H(t) = Ve^{\Delta t}L \]

Herein, \( V \) is a matrix of dimension \((N_o, 2N)\), containing as columns the mode shape coefficients at the \( N_o \) response locations of interest. As indicated in Section 1.3.3, \( L \) is a matrix of dimension \((2N, N_i)\) containing as rows the modal participation factors. Since the Laplace transform is a linear and homogeneous transform, the Laplace transform of \( H(t) \) equals,

\[ (54) \quad H(s) = V[La(e^{\Delta t})]L \]

\[ H(s) = V[sI - \Delta]^{-1}L \]

With the upper residuals \( U \), introduced in the preceding,
this partial fraction expansion is modified to,

(55) \[ H(s) = V[sI - \Lambda]^{-1}L + U \]

Equation (50) can be written in an equivalent linearized form ([86], pp. 11-15),

(56) \[ [sI - A]E(s)[H(s) - U] = B(s) \]

with,

\[
A = \begin{bmatrix}
-A_1 & -A_2 & \cdots & -A_{2p} \\
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & I & 0
\end{bmatrix}
\]

(57)

\[
E(s) = \begin{bmatrix}
s^{2p-1}I \\
s^{2p-2}I \\
\vdots \\
I
\end{bmatrix}, \quad B(s) = \begin{bmatrix}
s^{2p-1}B_1 + \cdots + B_{2p} \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

The matrix \( A \), defined above, is also known as the companion matrix for the matrices \( A_1, \ldots, A_{2p} \) of the polynomial defined by Equation (52). Assuming that every repeated eigenvalue of \( A \) has a full complement of independent eigenvectors, there exists a similarity transformation, represented by \( \tilde{V} \), that transform \( A \) to a diagonal form,
\[ \tilde{V}^{-1} \tilde{A} \tilde{V} = \Lambda \]

\[ \tilde{A} \tilde{V} = \tilde{V} \Lambda \]

Herein \( \Lambda \) is a diagonal matrix of dimension \((2N, 2N)\) with the eigenvalues of \( A \), and \( \tilde{V} \) is a matrix of dimension \((2N, 2N)\) with the corresponding right eigenvectors. From the structure of \( A \), it also follows that,

\[ \tilde{V} = \begin{bmatrix} V \Lambda^{2p-1} \\ V \Lambda^{2p-2} \\ \vdots \\ V \end{bmatrix} \]

The eigenvalues \( \Lambda \) and eigenvectors \( V \), defined in above equations, will be identified with the pole values and mode shapes. Premultiplying both sides of Equation (56) by \( \tilde{V}^{-1} \), and using the property expressed by Equation (58), it follows that,

\[ [sI - \Lambda] \tilde{V}^{-1} E(s) [H(s) - U] = \tilde{V}^{-1} B(s) \]

Substituting \( H(s) \) in this equation by \( R(s) \) from Equation (55), then Equation (60) becomes,

\[ [sI - \Lambda] \tilde{V}^{-1} V(s) [sI - \Lambda]^{-1} L = \tilde{V}^{-1} B(s) \]

with,

104
\[
\begin{bmatrix}
 s^{2p-1}v \\
 s^{2p-2}v \\
 \vdots \\
 v
\end{bmatrix}
\]

(62) \[ V(s) = \]

The only unknown in Equation (61) is \( L \), the matrix of modal participation factors. \( L \) can be identified from an analysis of Equation (61) in the proximity of the poles \( \lambda_i \). Hereto, let this equation succinctly be written as,

(63) \[ AL = \tilde{V}^{-1}B(s) \]

with,

(64) \[ a_{i,j} = (s - \lambda_i)(s - \lambda_j)^{-1}[\tilde{V}^{-1}]_i[V(s)]_j \]

Let \( s \) approach the pole \( \lambda_i \). It follows from the expression for the coefficients \( a_{i,j} \) that all elements in row \( i \) of \( A \) are zero, except for element \( a_{i,i} \) that is yet undefined,

(65) \[ a_{i,i} = \lim_{s \to \lambda_i} (s - \lambda_i)(s - \lambda_i)^{-1} \lim_{s \to \lambda_i} [\tilde{V}^{-1}]_i[V(s)]_i \]

From the definition of \( V(s) \), by Equation (62), it follows that the second limit in Equation (65) is exactly one. Applying De l'Hôpital's rule, the first limit is also seen to converge to one. From Equation (63) it then follows that row \( i \) of \( L \) equals row \( i \) of \( \tilde{V}^{-1}B(s) \), evaluated at the pole \( \lambda_i \),

(66) \[ [L]_i = \lim_{s \to \lambda_i} [\tilde{V}^{-1}B(s)]_i, \quad i = 1 \ldots 2N \]
Similarly, Equations (41) and (46) can be written in partial fraction form,

\begin{align*}
(67) \quad \{y\} &= V[sI - \Lambda]^{-1}L\{x\} + U\{x\} \\
(68) \quad y &= V[sI - \Lambda]^{-1}L + U
\end{align*}

In the latter equation however, L is not calculated using Equation (66) but by using the following equations that can be derived completely analogous,

\begin{align*}
(69) \quad [L]_i &= \lim_{s \to \lambda_i} [\bar{\gamma}^{-1}C(s)]_i, \quad i = 1 \ldots 2N \\
\text{with,} \\
(70) \quad C(s) &= \begin{bmatrix}
    s^{2p-1}c_1 + \ldots + c_{2p} \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\end{align*}
2.1.3 Time Domain Direct Parameter Models

In Section 2.1.1 it was proved that the equations of motion of a lumped parameter viscous damped mechanical structure, with $N$ degrees of freedom and response observed at $N_0$ response locations for force input at $N_1$ locations, can be represented by the matrix differential equation, defined by Equation (18), of dimension $(N_0, N_0)$ and order $2p$, defined by Equation (4). Also, from the results of the study of the canonical response expressions of a viscous damped mechanical structure in Section 1.3, it can be concluded that:

1. $H(t)$, the matrix of dimension $(N_0, N_1)$ with the impulse response functions at $N_0$ response locations for $N_1$ reference locations, can be represented by an expansion into the eigenmodes, as defined by Equations (1.3.30),

$$H(t) = Ve^{\Lambda t}L$$

$\Lambda$ is a diagonal matrix of dimension $(2N, 2N)$ with the pole values on the diagonal. $V$ is the matrix of dimension $(N_0, 2N)$ with the mode shape coefficients at the $N_0$ response locations for the $2N$ mode shapes, and $L$ is the matrix of dimension $(2N, N_1)$ with the modal participation factors for the $N_1$ reference locations. As argued in Section 1.3, the partial fraction expansion of $H(t)$ by Equation (71) requires
that every repeated pole value $\lambda_i$ has a full complement of mode shapes.

2. The forced response of a system, initially, i.e. at time zero, at rest, was defined by Equation (1.3.31), as,

$$(72) \quad \{y(t)\} = \int_0^t H(t-\tau)\{x(\tau)\} \, d\tau$$

The vector $\{y(t)\}$ has $N_o$ elements, representing the response, and the vector $\{x(t)\}$ has $N_i$ elements representing the force input. From Equation (1.3.17a) and (1.3.29), it follows that the forced response of a stable system is also described by Equation (72), after the contribution by initial disturbances is damped out. Consequently, the steady state response is eventually also described by Equation (72).

3. $[y(t)]$, the matrix of dimension $(N_o, N_i)$ with the free decay response at $N_o$ response locations for $N_i$ sets of initial conditions is derived, using Equation (1.3.19a) and (1.3.29), as,

$$(73) \quad [y(t)] = Ve^{\Lambda t}[g_o]$$

In this Equation, $V$ and $\Lambda$ are as defined for $H(t)$ by Equation (71). $[g_o]$ is a matrix of dimension $(2N,N_i)$ with the $N_i$ sets of initial conditions. Hence, Equation (73) is formally equal to Equation (71).
In this section, a finite difference equation will be discussed, that is an approximation for the matrix differential equation defined by Equation (18) and is usable for estimation purposes on uniformly sampled time domain data. The proposed finite difference equation represents an exact recursive relation for the uniformly sampled values of $H(t)$, defined by Equation (71). Consequently, using Equations (72) and (73) it can be shown that the same finite difference relation can describe the uniformly sampled forced and free decay response.

Let the uniformly sampled time be represented by $k\Delta t$, $\Delta t$ being the time increment and $k$ a positive integer, and let $H_k$ be a diagonal matrix of dimension $(N_1, N_1)$ with unit impulse sample functions,

$$ R_0 = (\Delta t)^{-1} I $$

$$ R_k = 0 \quad , \quad k \neq 0 $$

Representing the values of the uniformly sampled impulse response functions $H(k\Delta t)$ by $H_k$, a finite difference equation for $H_k$ can be justified to be in the form of a following autoregressive moving-average equation,

$$ [I - A_1q^{-1} - A_2q^{-2} - \ldots - A_{2p}q^{-2p}]H_k $$

$$ = [B_0 + B_1q^{-1} + \ldots + B_{2p-1}q^{-2p+1}]R_k $$

with,
(76) \[ H_k = v e^{k \Delta t_L} , \quad k \geq 0 \]

Herein \( p \) is related to \( N \) and \( N_0 \) by Equation (4) and the symbol \( q^{-1} \) designates the backshift operator. The matrices \( A_1, ..., A_{2p} \) are all of dimension \((N_0, N_0)\) and differ from the matrices in the left side of Equation (18) for which the same notation is used. The matrices \( B_0, ..., B_{2p-1} \) are all of dimension \((N_0, N_1)\) and differ also from the matrices in the right side of Equation (18).

The order \( 2p \) of the autoregressive part of Equation (75) is justified by the fact that \( H_k \) represents uniformly sampled impulse response functions of a \( N \) degree of freedom mechanical system. This requires that the characteristic equation of the above finite difference equation, defined by Equation (77) below, is a polynomial equation of order \( 2N \) in \( q^{-1} \) (considering the operator \( q^{-1} \) as a variable).

(77) \[ \left| I - A_1 q^{-1} - A_2 q^{-2} - ... - A_{2p} q^{-2p} \right| = 0 \]

Since the matrices \( A_1, ..., A_{2p} \) are of dimension \((N_0, N_0)\), Equation (77) represents an equation of order \( 2pN_0 \) in \( q^{-1} \), or using Equation (4) of order \( 2N \) in \( q^{-1} \). From an analysis of the Z-transform image of the above finite difference equation, it will be proved later that the \( 2N \) pole values \( \lambda_i \) and corresponding mode shapes \( \{V\}_i \) in the partial fraction expansion of \( H(t) \) can be derived from the coefficients.
A_1, \ldots, A_{2p}.

To justify the order (2p-1) of the moving-average part of Equation (75), it is helpful to express the finite difference equation by its equivalent set of recursive relations,

\[ H_0 = B_0 \]
\[ H_1 = A_1 H_0 + B_1 \]
\[ H_2 = A_1 H_1 + A_2 H_0 + B_2 \]
\[ \vdots \]
\[ H_{2p-1} = A_1 H_{2p-2} + \cdots + A_{2p-2} H_0 + B_{2p-1} \]
\[ H_k = A_1 H_{k-1} + \cdots + A_{2p} H_{k-2p}, \quad k \geq 2p \]

(79)

In Equations (78) use is made of the property that, for causal systems, \( H_k \) equals 0 when \( k < 0 \). These equations allow the recursive calculation of \( H_0, \ldots, H_{2p-1} \), that consecutively become starting values for the recursive calculation of \( H_k \) by Equation (79) for \( k \geq 2p \). Note that the coefficients \( B_0, \ldots, B_{2p-1} \) of the moving-average part only appear in the \((2p-1)\) relations expressed by Equations (78) but not in the recursive relation expressed by Equation (79).

These recursive relations define explicitly the series \( H_k \), that is a solution of the finite difference relation defined by Equation (75). From the factored form of \( H_k \), as by
Equation (76), with \( \lambda_i \) and \( \{v\}_i \) being determined by the autoregressive part of Equation (75), it follows that for every mode \( N_i \) coefficients, expressed by \( \{L\}_i \), remain to be determined. In total this represents \( 2NN_i \) coefficients, exactly as many as there are elements in the \( 2p \) matrices \( B_0, ..., B_{2p-1} \). Therefore, a moving-average part of order \( (2p-1) \) will in general be required. With Equations (78) \( H_0, ..., H_{2p-1} \) are calculated recursively using explicitly the coefficients \( B_0, ..., B_{2p-1} \) of the moving average part and fixing implicitly the \( 2NN_i \) initial conditions that are represented by \( L \) in Equation (76).

An explicit relation between the modal participation factors \( L \) and \( H_0, ..., H_{2p-1} \) can be found using the partial fraction expansion of \( H_k \):

\[
\begin{bmatrix}
V e^\lambda (2p-1) \Delta t \\
V e^\lambda (2p-2) \Delta t \\
\vdots \\
V
\end{bmatrix}
L =
\begin{bmatrix}
H_{2p-1} \\
H_{2p-2} \\
\vdots \\
H_0
\end{bmatrix}
\]

Concisely written as,

\[
\bar{V}L = \bar{H}
\]

This equation is uniquely solvable for \( L \) when the coefficient matrix \( \bar{V} \) is invertible. From an analysis of the Z-transform image of the difference equation defined by Equa-
tion (75), this will further on be proved to be so when
every repeated pole value $\lambda_i$ has a full complement of inde-
pendent mode shapes.

Consider the matrix $H(t)^t$ that, from Maxwell-Betti recipro-
city, represents the impulse response functions at $N_i$
response locations for $N_o$ reference locations. Therefore, a
finite difference relation for $H_k^t$ is of the form,

\begin{equation}
[I - A_1q^{-1} - A_2q^{-2} - \ldots - A_{2p}q^{-2p}] H_k^t = [B_0 + B_1q^{-1} + \ldots + B_{2p-1}q^{-2p+1}] R_k
\end{equation}

The matrices $A_1, \ldots, A_{2p}$ are of dimension $(N_i, N_i)$, the
matrices $B_0, \ldots, B_{2p-1}$ if dimension $(N_i, N_o)$. The matrix $R_k$
is of dimension $(N_o, N_o)$, with unit sample functions on the
diagonal. The order $p$ is now selected from following
equation,

\begin{equation}
N = pN_i
\end{equation}

Since in very many practical situations, $N_o$ is a lot larger
than $N_i$, the order $p$ of the model defined by Equation (82)
is larger as compared to the order of the model defined by
Equation (75).

Representing the uniformly sampled values $\{x(k\Delta t)\}$ and
$\{y(k\Delta t)\}$ by $\{x_k\}$ and $\{y_k\}$ respectively, and approximating
the convolution integral in Equation (72) by a summation, an
expression for $\{y_k\}$ becomes,
(84) \( \{y_k\} = \Delta t \sum_{i=0}^{k} H_{k-i}\{x_i\} \)

\( H_k \) is however a solution of the finite difference equation, given by Equation (75). Using \( A^{2p}(q^{-1}) \) to represent the autoregressive operator and \( B^{2p-1}(q^{-1}) \) the moving average operator, this finite difference equation can succinctly be written as,

(85) \( H_k = \left[ \frac{B^{2p-1}(q^{-1})}{A^{2p}(q^{-1})} \right] R_k \)

Substituting \( H_k \) in Equation (84) by \( H_k \) from Equation (85), and using the definition of \( R_k \) expressed by Equation (74), it follows that,

(86) \( \{y_k\} = \left[ \frac{B^{2p-1}(q^{-1})}{A^{2p}(q^{-1})} \right]\{x_k\} \)

or,

(87) \[
[I - A_1q^{-1} - \ldots - A_{2p}q^{-2p}] \{y_k\} = [B_0 + B_1q^{-1} + \ldots + B_{2p-1}q^{-2p+1}] \{x_k\}
\]

Finally, representing the uniformly sampled values \([y(k\Delta t)]\) by \([y_k]\), it follows directly from the similarity between Equations (71) and (73) that a valid finite difference equation for \([y_k]\) is given by,
\[(88) \quad [I - A_1q^{-1} - \ldots - A_{2p}q^{-2p}][y_k] = [C_0 + C_1q^{-1} + \ldots + C_{2p-1}q^{-2p+1}] N_k \]

The coefficients \(C_0, \ldots, C_{2p-1}\) are all matrices of dimension \((N_o, N_i)\). In both Equation (87) and (88), \(p\) is defined by Equation (4).

Using \([y_k]^t\), the following finite difference relation can be considered,

\[(89) \quad [I - A_1q^{-1} - \ldots - A_{2p}q^{-2p}][y_k]^t = [C_0 + C_1q^{-1} + \ldots + C_{2p-1}q^{-2p+1}] R_k \]

The matrices \(A_1, \ldots, A_{2p}\) are of dimension \((N_i, N_i)\) and the matrices \(C_0, \ldots, C_{2p-1}\) of dimension \((N_i, N_o)\). The order \(p\) is as defined by Equation (83).

As an aside, it is worth mentioning at this point that an attempted approximation of the time derivatives \((d/dt)\) in Equation (18) by the finite difference operator \(\nabla\),

\[(90) \quad \nabla = 1 - q^{-1} \]

will yield an autoregressive moving average model, however with moving-average part of order \((2p-2)\) rather than \((2p-1)\).
(92) \[ I - A_1q^{-1} - \ldots - A_{2p}q^{-2p}\{y_k\} = [B_0 + B_1q^{-1} + \ldots + B_{2p-2}q^{-2p+2}\{x_k\} \]

Note that the matrix coefficients in Equation (92) do not equal the ones in Equation (91). The same symbols have been used for notational convenience. Equation (92) equals Equation (87) when in this latter equation \(B_{2p-1}\) equals 0. Therefore, this latter equation, derived to represent explicitly a uniform sampled sequence of a solution of Equation (18), is to be preferred for estimation purposes.

The Equations (75), (87) and (88) represent the time domain equivalent of the frequency domain models defined by Equations (38), (43) and (48) respectively. The matrix coefficients in these equations can be estimated from sampled time series. The estimation schemes and conditions for which the parameters can be uniquely estimated from a given data set will be discussed in Section 2.3.

In the remainder of this section, a partial fraction expansion of the Z-transform image of Equation (75), (87) and (88) will be derived. The parameters in this partial fraction expansion are identifiable with the global modal parameters of the structure as defined in Section 1.3.3. Hereto, it is convenient to rewrite Equation (76) more explicitly,

(93) \[ H_k = Vz^k \bar{U}_k^L, \quad z = e^{LA} \]
or,
\[ H_k = \sum_{i=1}^{2N} \{V\}_i z_i^k u_k \{L\}_i, \quad z_i = e^{\lambda_i \Delta t} \]

\( \mathbf{u}_k \) is a diagonal matrix of dimension \((2N, 2N)\) with sampled unit step functions on the diagonal,

\[ \mathbf{u}_k = 0, \quad k < 0 \]
\[ \mathbf{u}_k = I, \quad k \geq 0 \]

(94)

The series \( H_k \), represented by Equation (93), is defined for all integers \( k \). The Z-transform of \( H_k \), represented by \( H(z) \), is given by ([87], Chapter 2),

(95) \[ H(z) = Z^{-1}(H_k) = V[I - z^{-1}I]^{-1}L \]

or,
\[ H(z) = \sum_{i=1}^{2N} \{V\}_i (1 - z^{-1}z_i)^{-1} \{L\}_i \]

\(|z| > \max(|z_i|)\) and \(|z_i| < 1\) for \( i = 1 \ldots 2N \)

The Z-transform of the series \( D_k \) defined by Equation (74) is given by,

(96) \[ Z^{-1}(D_k) = (\Delta t)^{-1}I \]

Using this property, the Z-transform image of the finite difference equation defined by Equation (75) is,
\begin{align}
(97) \quad [z^{2p}I - z^{2p-1}A_1 - \ldots - A_{2p}] H(z) \\
\quad = (\Delta t)^{-1}[z^{2p}B_0 + z^{2p-1}B_1 + \ldots + zB_{2p-1}] \\
\text{or,}
(98) \quad H(z) = \frac{[z^{2p}B_0 + z^{2p-1}B_1 + \ldots + zB_{2p-1}]}{\Delta t [z^{2p}I - z^{2p-1}A_1 - \ldots - A_{2p}]}
\end{align}

This equation is a rational matrix polynomial expression for the impulse response function matrix in the $z$ domain; evaluated for values of $z$ along the unit circle, this equation expresses the matrix of frequency response functions as a rational matrix polynomial ([87], pp. 89).

A partial fraction expansion will be derived for this equation under the assumption that every repeated eigenvalue of the characteristic polynomial,

\begin{equation}
(99) \quad [z^{2p}I - z^{2p-1}A_1 - \ldots - A_{2p}]
\end{equation}

has a full complement of independent eigenvectors. Hereto, Equation (97) is first written in an equivalent linearized form,

\begin{equation}
(100) \quad [zI - A]E(z)H(z) = B(z)
\end{equation}

with,
\[
A = \begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} \\
I & 0 & \cdots & 0 \\
0 & I & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & I & 0
\end{bmatrix}
\]

(101)

\[
E(z) = \begin{bmatrix}
z^{2p-1}I \\
z^{2p-2}I \\
\vdots \\
I
\end{bmatrix}, \quad B(z) = (\Delta t)^{-1}\begin{bmatrix}
z^{2p}B_0 + \cdots + zB_{2p-1} \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

The matrix \( A \) is the companion matrix of the matrix coefficients \( A_1, \ldots, A_{2p} \) in the polynomial defined by Equation (99). Assuming that every repeated eigenvalue of this polynomial has a full complement of eigenvectors, there exists a similarity transformation, represented by \( \tilde{V} \), that transforms \( A \) to a diagonal form,

\[
\tilde{V}^{-1}A\tilde{V} = \mathbf{Z}
\]

(102)

\[
A\tilde{V} = \tilde{V}\mathbf{Z}
\]

In these equations, \( \mathbf{Z} \) is a diagonal matrix of dimension \((2N,2N)\) with the eigenvalues of \( A \) on the diagonal, and \( \tilde{V} \) is a matrix of dimension \((2N,2N)\) with the corresponding eigenvectors. From the structure of \( A \) it also follows that,
\[
\bar{V} = \begin{bmatrix}
\bar{V}x^{2p-1} \\
\bar{V}x^{2p-2} \\
. \\
. \\
. \\
\bar{V}
\end{bmatrix}
\]

Premultiplying both sides of Equation (100) with \(\bar{V}^{-1}\) and using the property expressed by Equation (102), it follows,

\[
(104) \quad [zI - \bar{X}]\bar{V}^{-1}E(z)H(z) = \bar{V}^{-1}B(z)
\]

Substituting \(H(z)\) in this equation by \(H(z)\) from Equation (95) and identifying \(\bar{X}\) and \(\bar{V}\) in Equation (102) and (103) with \(\bar{X}\) and \(\bar{V}\) in Equation (95), then,

\[
(105) \quad [zI - \bar{X}]\bar{V}^{-1}V(z)[I - z^{-1}\bar{X}]^{-1}L = \bar{V}^{-1}B(z)
\]

with,

\[
\bar{V}(z) = \begin{bmatrix}
z^{2p-1} \bar{V} \\
z^{2p-2} \bar{V} \\
. \\
. \\
. \\
\bar{V}
\end{bmatrix}
\]

The modal participation factors, \(L\), are the only unknowns in Equation (105), and can be calculated from an analysis of the behavior of Equation (105) in the neighborhood of the poles \(z_i\) of that equation. This analysis completely parallels the analysis outlined by Equations (63), (64), (65) and (66) in Section 2.1.2, and yields,
\[(107) \quad \left[L\right]_i = \lim_{z \to z_i} (z^{-1} [\tilde{V}^{-1} B(z)]_i) \quad , \quad i = 1 \ldots 2N \]

As mentioned before, \( L \) can also alternatively be calculated using Equation (80).

The \( Z \)-transform image of Equation (82) gives following rational matrix polynomial,

\[(108) \quad H(z)^t = \frac{[z^{2p} B_0 + z^{2p-1} B_1 + \ldots + z B_{2p-1}]}{\Delta t [z^{2p} I - z^{2p-1} A_1 - \ldots - A_{2p}]} \]

From Equation (95) also follows,

\[(109) \quad H(z)^t = L^t [I - z^{-1} \tilde{X}]^{-1} \tilde{V}^t \]

Completely analogous to the preceding it can be proved that \( L, \tilde{X} \) and \( \tilde{V} \) can be calculated from following expressions.

\[(110) \quad A \tilde{L} = \tilde{L} \tilde{X} \]

\[ \begin{bmatrix} L t z^{2p-1} \\ L t z^{2p-2} \\ \vdots \\ L t \end{bmatrix} \]

\[(111) \quad \tilde{L} = \begin{bmatrix} L t z^{2p-1} \\ L t z^{2p-2} \\ \vdots \\ L t \end{bmatrix} \]

\[(112) \quad [\tilde{V}^t]_i = \lim_{z \to z_i} (z^{-1} [\tilde{L}^{-1} B(z)]_i) \quad , \quad i = 1 \ldots 2N \]

In these equations, \( A \) and \( B(z) \) are of the form defined in Equation (101).
Let the Z-transform of \{x\} and \{y\} be represented by \{X\} and \{Y\}. The Z-transform images of Equation (87) and (88) become,

\[
(113) \quad [z^{2p}I - z^{2p-1}A_1 - \ldots - A_{2p}]{Y}
= [z^{2p}B_0 + z^{2p-1}B_1 + \ldots + zB_{2p-1}]{X}
\]

\[
(114) \quad [z^{2p}I - z^{2p-1}A_1 - \ldots - A_{2p}]{Y}
= z^{2p}C_0 + z^{2p-1}C_1 + \ldots + zC_{2p-1}
\]

The respective partial fraction expansions are,

\[
(115) \quad {Y} = V[I - z^{-1}x]^{-1}L{X}
\]

\[
(116) \quad {Y} = V[I - z^{-1}x]L
\]

In this latter equation, L is not defined by Equation (107) but by,

\[
(117) \quad [L]_1 = \lim_{z \to z_1} \left( z^{-1} [\tilde{V}^{-1}C(z)]_1 \right)
\]

with,

\[
(118) \quad C(z) = (\Delta t)^{-1}
\begin{bmatrix}
  z^{2p}C_0 + \ldots + zC_{2p-1} \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]

When, for the free decay response, Equation (89) is used instead of Equation (88), then \(X\) and \(L\) in Equation (116) will be calculated from Equation (110) and (111), and \(V\)
from,

\[ \lim_{z \to z_1} (z^{-1}L^{-1}C(z)) \]

2.1.4 Characteristics of Modal Parameters Calculated from the Direct Parameter Models

Before proceeding with the discussion of estimation methods for parameters in the direct parameter models, it is worth qualifying further their use in the calculation of the modal parameters of a structure. Reference will be made to the relations between the modal parameters and the parameters in the time domain models, developed in Section 2.1.3.

An estimate for the modal parameters is calculated from the parameters in the time domain models, essentially using equations like Equation (102), (103) and (107). As will be discussed in the next sections, the parameters in the time domain models can be identified using all available measurement data simultaneously, so that a global estimate of modal parameters is obtained.

For a structure with \( N \) modes, excited at \( N_1 \) force input locations and response measured at \( N_o \) locations, a total of \( 2N(N_1+N_o) \) real coefficients make up the parameters in the
direct parameter model. On the other hand, there will be $2N$ pole values and $2NN_0N_i$ corresponding residue coefficients, in total $4N(1+N_0N_i)$ real parameters, in the modal model. The apparent savings are a direct result of the calculation of the modal parameters from the direct parameter model. The pole values and mode shapes are calculated as eigenvalues and eigenvectors of a real matrix, and therefore will occur in complex conjugate pairs. The same is true for the modal participation factors that are calculated using Equation (107), since the coefficients of $B(z)$, defined by Equation (101), are real and the eigenvalues and eigenvectors occur in complex conjugate pairs. The residue matrices are found in factored form as,

$$R_i = [V_i] [L_i], \quad i = 1 \ldots 2N$$

Since $[V_i]$ and $[L_i]$ occur in complex conjugate pairs, the residue matrices $R_i$ occur in complex conjugate pairs. Also, every column of a residue matrix $R_i$ describes the same mode shape $[V_i]$, that is, the $N_0N_i$ coefficients in a residue matrix are found as combinations of $(N_0+N_i)$ coefficients.

When the structure satisfies the Maxwell-Betti reciprocity, then it follows from the discussion in Section 1.3.4 that for every combination of two input locations $i_1$ and $i_2$, $1 \leq i_1, i_2 \leq N_i$, corresponding to two response locations $o_1$ and $o_2$, $1 \leq o_1, o_2 \leq N_o$, then the following relation should
be satisfied for the combined residue matrices $\mathbf{R}_{ci}$, defined
in Section (1.3.3),

$$ (121) \quad (\mathbf{R}_{ci})_{o_1,i_2} = (\mathbf{R}_{ci})_{o_2,i_1} $$

For the residue matrices of non-repeated modes, it follows
also that,

$$ (122a) \quad (\mathbf{R}_i)_{o_1,i_2} = (\mathbf{R}_i)_{o_2,i_1} $$

or,

$$ (122b) \quad \mathbf{v}_{o_1,i_1,i_2} = \mathbf{v}_{o_2,i_1,i_1} $$

No constraints have however been enforced on the parameters
in the direct parameter models so that the above relations
between mode shapes and modal participation factors should
be fulfilled. How close such relations are satisfied will
entirely be determined by how well the measurement data
itself satisfies the reciprocity principle. A reciprocity
correlation number can be defined between both sides of
Equation (121) or (122), and be used to verify how well the
reciprocity relations are fulfilled. As will be discussed in
Section 2.4, such a correlation number can help distinguish
structural modes from noise modes and computational modes.
In the latter section, methods will also be discussed to
estimate modal mass, as defined in Section 1.3.6, from
multiple columns in the residue matrices.

It is theoretically possible to identify repeated modes that have a full complement of independent mode shapes. As mentioned before, the pole values and mode shapes can be identified as the eigenvalues and eigenvectors of the matrix $A$, defined by Equation (101). $A$ is the companion matrix of the matrix coefficients $A_1, \ldots, A_2p$ in the polynomial defined by Equation (99). The matrices $A_i$ being of dimension $(N_0, N_0)$, $A$ can theoretically have eigenvalues of multiplicity $N_0$ that still have a full complement of $N_0$ independent eigenvectors.

The matrices $A_1, \ldots, A_{2p}$ are to be identified from the data. For example, let the data be impulse response functions at $N_0$ response locations for $N_1$ reference locations, so that $A_i$ are the parameters in the model defined by Equation (75).

As indicated by Equation (76), the impulse response functions are a weighted superposition of modes. Let the matrix of pole values $\Lambda$ be partitioned like in Equation (1.3.9). The contribution of a mode with pole value $\lambda_i$ and multiplicity $m_i$ to the impulse response functions equals,

$$V_i e^{\Lambda_i k \Delta t_{L_i}}$$

Herein, $\Lambda_i$ is a diagonal matrix of dimension $(m_i, m_i)$ with
the pole values $\gamma_i$ on the diagonal, $V_i$ represents a matrix of dimension $(N_0, m_i)$ with the $m_i$ independent mode shapes and $L_i$ is a matrix of dimension $(m_i, N_i)$ with modal participation factors.

Since $V_i$ is of dimension $(N_0, m_i)$, the number of independent vectors cannot be larger than $N_0$, in agreement with what was observed above. It is proved next that to have a contribution by $m_i$ independent modes to the impulse response functions, also the matrix $L_i$ needs to be of rank $m_i$.

Consider the matrix product $V_i L_i$. Assuming that the rank of $V_i$ equals $m_i$, the rank of this product equals the rank of $L_i$. Let the rank of $L_i$ equal $r$, $r < \min(m_i, N_i)$. Then $V_i L_i$ can alternatively be expressed as,

$$ (124) \quad V_i L_i = V'_i L'_i $$

$V'_i$ represents a matrix of dimension $(N_0, r)$ with as columns $r$ independent combinations of the $m_i$ columns of $V_i$. $L'_i$ is a matrix of dimension $(r, N_i)$ with the corresponding modal participation factors. The contribution of the mode with multiplicity $m_i$ to the impulse response functions then can be expressed as the contribution of a mode with multiplicity $r$,

$$ (125) \quad V'_i e^{i \gamma_i t} L'_i $$

The matrix $\Lambda'_i$ is a diagonal matrix of dimension $(r, r)$ with
$\lambda_i$ on the diagonal. Therefore, a mode that may have a multiplicity of $m_i$ only appears in the impulse response functions as a mode with multiplicity $r$. Only when $r$ equals $m_i$, that is when the mode is excited from the $N_i$ reference locations in $m_i$ independent combinations, can the mode be observed from the data as a mode with multiplicity $m_i$. It follows that $N_i$ should at least equal $m_i$, and that a mode with multiplicity $m_i$, $m_i > \min(N_0, N_i)$, can never be identified with still $m_i$ independent mode shapes.
2.2 Direct Parameter Model Identification in the Frequency Domain

In this section, estimation procedures are developed for the parameters in the models defined by Equation (2.1.38), (2.1.43) and (2.1.48). The least squares technique is used as pseudo-inverse procedure to derive equation systems that can be solved for least squares estimates of the model parameters.

The data quantities \( \{X_k\}, \{Y_k\}, H_k \) and \( Y_k \), that are used to estimate the model parameters, will be contaminated with measurement noise. This noise can have the nature of a random signal, can be composed of systematic signals causing for example a trend or drift of the measured time series or a systematic phase shift on the corresponding frequency spectra, or can be a combination of both. In the developed estimation procedures, it is assumed that the noise can appropriately be accounted for by including complementary degrees of freedom equivalent to \( N_c \) modes in the direct parameter model. Therefore, parameters in a model for a mechanical structure with \( N_m \) modes will be estimated on the
data. $N_m$ is related to $N_c$, to $N$, the effective number of modes of the structure under examination, and to $N_o$, the number of generalized coordinates, by following equations,

1. $N_m = N_c + N$
2. $N_m = pN_o$, $p$ is an integer

As expected and verified experimentally, the number of complementary degrees of freedom needs to be larger for higher levels of noise.

Eventually several samples may be available for the frequency spectra of the input and output signals, $\{X_k\}$ and $\{Y_k\}$. To describe different samples, an additional index is introduced and the notation for $\{X_k\}$ and $\{Y_k\}$ is altered to $\{X_k\}_i$ and $\{Y_k\}_i$. The number of available samples will be indicated by $N_s$. All columns $\{X_k\}_i$ and $\{Y_k\}_i$, $i = 1, \ldots, N_s$, will compactly be represented by two matrices $X_k$ and $Y_k$ of dimension $(N_o, N_s)$ and $(N_i, N_s)$ respectively.

2.2.1 Conditions for Uniqueness of the Parameters

Before developing the estimation procedures and equations for the parameters in the frequency domain models, some general comments will be made on conditions for which the parameters in these models are uniquely described by the
data. Such conditions help to determine when the estimation equations to be solved for the parameters are full rank and have a unique set of parameters as solution.

Consider the model defined by Equation (2.1.38), compactly representable as,

\[
A^2P(s_k) Y_k = B^2P(s_k) X_k , \quad s_k = j\omega_k \\

k = 1 \ldots N_f , \quad i = 1 \ldots N_s \\
A^{2P}(s) = s^{2P} I + s^{2P-1} A_1 + \ldots + A_{2P} \\
B^{2P}(s) = s^{2P} B_0 + \ldots + B_{2P}
\]

The prime notation that was used with the coefficients \(B_0, \ldots, B_{2P}\) in Equation (2.1.38) is omitted to simplify the notation. The order \(p\) is as defined by Equation (2). The vectors \(X_k\) and \(Y_k\) are complex. The matrices \(A_i\) and \(B_i\) are real and will be forced to be real in the estimation procedures that are developed.

Two necessary conditions for uniqueness of the coefficients in \(A^{2P}(s)\) and \(B^{2P}(s)\) can be proved to be,

\[
(4) \quad [(X) R \ (X)^{-1}] \text{ has rank } N_i \\
(5) \quad [(Y) R \ (Y)^{-1}] \text{ has rank } N_o
\]

with,

\[
(6) \quad X = [\{X_1\} \ldots \{X_{N_f}\} N_s]
\]
(7) \( Y = \{Y_1\}_1 \ldots \{Y_{N_f}\}_N \) 

To prove that the conditions expressed by above equations are necessary, use is made of following property. Let \( A \) be a real matrix of dimension \((m,n)\) with rank \( r \), \( r \leq \min(m,n) \). There exists an invertible real matrix \( S \) of dimension \((m,m)\) with the property that,

(8) \( SA = A \), \( S \neq I \)

The proof of this property follows by construction of \( S \). Let the rows of \( A \) be represented by \([A]_1, \ldots, [A]_m\). Since \( r \) is smaller than either \( m \) or \( n \), the rows are linearly dependent, indicating that \( m \) real numbers \( c_1, \ldots, c_m \), not all simultaneously zero, exist such that,

(9) \( c_1[A]_1 + \ldots + c_m[A]_m = 0 \)

Let \( c_1 \) be the first coefficient that is non-zero, and let the coefficients be normalized so that \( c_1 \) equals 1. Select \( S \) equal to \( I \), except for row \( i \) that is taken equal to,

(10) \( [S]_i = [c_1 \ldots (c_i+1) \ldots c_m] \)

It is easily verified that the rows of \( S \) are independent, i.e. \( S \) is invertible, and that \( S \) has the property expressed by Equation (8).

Hence, when the condition expressed by Equation (4) is not satisfied, at least one invertible real matrix \( S_i \) of
dimension \((N_i, N_i)\) exists such that,

\[(11) \quad S_i[(X)_R (X)_I] = [(X)_R (X)_I], \quad S_i \neq I\]

Similarly, when the condition expressed by Equation (5) is not fulfilled, an invertible real matrix \(S_o\) of dimension \((N_o, N_o)\) exists with the property,

\[(12) \quad S_o[(Y)_R (Y)_I] = [(Y)_R (Y)_I], \quad S_o \neq I\]

Then the model defined by Equation (3) can alternatively be expressed as,

\[(13) \quad A'^2p(s_k)\{x_k\}_i = B'^2p(s_k)\{x_k\}_i, \quad s_k = j\omega_k\]

\[k = 1 \ldots N_f, \quad i = 1 \ldots N_s\]

\[A'^2p(s) = S_o^{-1}A^2p(s)S_o\]

\[B'^2p(s) = S_o^{-1}B^2p(s)S_i\]

Therefore, when the conditions expressed by Equation (4) and (5) are not simultaneously satisfied, the parameters in the model defined by Equation (3) are not unique.

Remark that \(A'^2p(s)\) is derived from \(A^2p(s)\) by a similarity transformation so that the eigenvalues and eigenvectors of \(A'^2p(s)\) and \(A^2p(s)\) are identical. \(A'^2p(s)\) and \(B'^2p(s)\) are also matrix polynomials with real coefficients, and the highest order matrix coefficient of \(A^2p(s)\) still equals I.

The conditions expressed by Equation (4) and (5) are also
sufficient. To argue this, recall the canonical response expression defined by Equation (2.1.67),

\[(14) \{Y_k\}_i = V[s_k I - \Lambda]^{-1}L\{X_k\}_i + U\{X_k\}_i, \quad s_k = j\omega_k\]

When the conditions expressed by Equation (4) and (5) are satisfied, it is easily verified that the only alternative canonical expression equals,

\[(15) \{Y_k\}_i = V'[s_k I - \Lambda]^{-1}L'\{X_k\}_i + U\{X_k\}_i\]

with,

\[(16) \quad V' = VS\]
\[(17) \quad L' = S^{-1}L\]

But \(\Lambda\) and \(V\), or \(V'\), are calculated as the eigenvalues and eigenvectors of the matrix polynomial \(A^2P(s)\) as outlined in Section 2.1.2 using Equation (2.1.58) and (2.1.59). Therefore an alternative matrix polynomial for \(A^2P(s)\) in the model defined by Equation (3), \(A'2P(s)\), has to be similar to \(A^2P(s)\),

\[(18) \quad A'2P(s) = S^{-1}A^2P(s)S_O\]

When the condition expressed by Equation (5) is satisfied however, then the only possible \(S_O\) in Equation (18) equals \(I\), which warrants the uniqueness of \(A^2P(s)\). \(L\) and \(U\) are calculated using the coefficients of the polynomial \(B^2P(s)\),
having obtained $\Lambda$ and $V$ as eigenvalues and eigenvectors of $A^2P(s)$, and using Equation (2.1.65) discussed in Section 2.1.2. The conditions expressed by Equations (4) and (5) being satisfied and having fixed the scaling of the eigenvectors $V$, the matrices $L$ and $U$ are uniquely defined. Together $L$ and $U$ represent $(2NN_i + N_0N_i)$ coefficients, equal to the total number of coefficients in the matrix polynomial $B^2P(s)$; it's coefficients therefore are uniquely defined.

In the above reasoning, it is implicitly assumed that the response $\{Y_k\}_i$ is related to the force input $\{X_k\}_i$ by a canonical expression of the form defined by Equation (14) with $N_m$ modes, including $N_c$ complementary modes to describe the noise. The uniqueness of the parameters in the model defined by Equation (3) will not be in question when the conditions specified by Equation (4) and (5) are satisfied, assuming that $N_m$ is not taken unreasonably large in comparison with $N$.

From the property that the rank of the product of a matrix with its transpose still equals the rank of the matrix, it follows that the conditions expressed by Equation (4) and (5) can alternatively be expressed as,

(19) $[(X)_R (X)_I][(X)_R (X)_I]^T$ has rank $N_i$
(20) $[(Y)_R (Y)_I][(Y)_R (Y)_I]^T$ has rank $N_o$
or,

\[(21) \quad (G_{XX})_R = \sum_{i=1}^{N_f} (G_{X_kX_k})_R \quad \text{has rank } N_i \]

\[(22) \quad (G_{YY})_R = \sum_{i=1}^{N_f} (G_{Y_kY_k})_R \quad \text{has rank } N_o \]

with,

\[(23) \quad G_{X_kX_k} = X_k^h X_k^h , \quad k = 1 \ldots N_f \]

\[(24) \quad G_{Y_kY_k} = Y_k^h Y_k^h , \quad k = 1 \ldots N_f \]

Note that to satisfy the conditions expressed by Equation (4) and (5), or alternatively Equation (21) and (22), it is at least mandatory that,

\[(25) \quad (N_sN_f)/2 \geq \max(N_i, N_o) \]

The condition expressed by Equation (4) for the input forces is fairly weak. Since the number of inputs, $N_i$, is generally small, it will only be in exceptional situations that their spectra will be linearly related at all frequencies. The condition expressed by Equation (5) for the responses is of more importance. This condition will not be satisfied when the response at some response locations is a linear combination of the response at the remainder locations. This may be uncommon when the total number of response locations, $N_o$, is small, in particular, smaller than $N_m$. As will be discussed in Section 2.2.3, it will however occur when $N_o$ is larger
than $N_m$, and a model like expressed by Equation (2.1.38) with $p$ equal to $i$ becomes a large order incomplete model as compared to a low order complete model.

Similarly, it can be derived that the parameters in the model for the free decay response, defined by Equation (2.1.43), will be uniquely described by the data, when following condition is satisfied,

(26) $[(Y)_R (Y)_f]$ has rank $N_0$

with,

(27) $Y = [Y_1 \cdots Y_{N_f}]$

Finally, the parameters in the model for the impulse responses, defined by Equation (2.1.48), are unique when,

(28) $[(H)_R (H)_f]$ has rank $N_0$

with,

(29) $H = [H_1 \cdots H_{N_f}]$

To satisfy the condition defined by Equation (26) or (28), given $N_1$ initial conditions or reference locations, it is required that,

(30) $(N_1N_f)/2 \geq N_0$
The interpretation for the conditions expressed by Equation (26) and (28) is similar to the interpretation given for the condition expressed by Equation (5).

Estimation schemes for the parameters in the frequency domain models, that result in full rank equation systems when the conditions expressed by Equation (4) and (5), (26) or (28) are satisfied, will be developed in following section. When the conditions expressed by the latter equations are not fulfilled, as will for example be the case when \( N_m \) is smaller than \( N_o \) and the model becomes a large order incomplete model, then it still will be possible to obtain full rank equation systems for an equivalent low order complete model after appropriate preprocessing of the data. Such preprocessing methods form the subject of Section 2.2.3.

2.2.2 Least Squares Estimates

2.2.2.1 Least Squares Problem Formulation

Consider the model defined by Equation (2.1.38), repeated in Equation (3). The factors in this equation can be reorganized to yield,
(31) \[ s^{2p-1}A_1 + \ldots + A_{2p}Y_k \]
\[- [s^{2p}B_0 + \ldots + B_{2p}]X_k = - Y_k \]
\[ s_k = j\omega_k, \quad k = 1, \ldots, N_f \]

\( X_k \) and \( Y_k \) are matrices of dimension \((N_i, N_s)\) and \((N_o, N_s)\) with \(\{X_k\}_i\) and \(\{Y_k\}_i\) as columns. The order \(p\) is as defined in Equation (2). All equations like Equation (31) can succinctly be represented by one global matrix equation,

\[
\begin{bmatrix}
A_1 & \ldots & A_{2p} & B_0 & \ldots & B_{2p}
\end{bmatrix}
\begin{bmatrix}
\ldots & s^{2p-1}Y_k & \ldots \\
\vdots & \vdots & \vdots \\
\ldots & Y_k & \ldots \\
\ldots & s^{2p}X_k & \ldots \\
\ldots & -X_k & \ldots \\
\ldots & -s^{2p}Y_k & \ldots
\end{bmatrix}
= \begin{bmatrix}
\ldots & -s^{2p}Y_k & \ldots
\end{bmatrix}, \quad s_k = j\omega_k
\]
\[ k = 1, \ldots, N_f \]

This equation represents a set of \(N_i N_s\) simultaneous linear equations with as unknowns the parameters \(A_1, \ldots, A_{2p}\) and \(B_0, \ldots, B_{2p}\). By including data at sufficient frequency points \(N_f\) and sufficient samples \(N_s\), the equation will be overdetermined but not necessarily consistent, which is indicated by the use of the approximation sign \(\approx\).

The parameters \(A_i\) and \(B_i\) are real, while the coefficient matrix and right hand side in Equation (32) in general are
complex. From equating real and imaginary part, the following equation with real coefficient matrix and right hand side is derived,

\[
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} & B_0 & B_1 & \cdots & B_{2p} \\
\omega_k^{2p-1} & \sigma_I^{2p-1}(y_k)_I & \omega_k^{2p-1} & \sigma_R^{2p-1}(y_k)_R & \cdots \\
\omega_k^{2p-2} & \sigma_R^{2p-2}(y_k)_R & \omega_k^{2p-2} & \sigma_I^{2p-2}(y_k)_I & \cdots \\
& \vdots & & \vdots & \\
\omega_k^{2p} & \sigma_I^{2p}(x_k)_I & \omega_k^{2p} & \sigma_R^{2p}(x_k)_R & \cdots \\
\omega_k^{2p-1} & \sigma_I^{2p-1}(x_k)_I & \omega_k^{2p-1} & \sigma_R^{2p-1}(x_k)_R & \cdots \\
& \vdots & & \vdots & \\
\omega_k & \sigma_I & \omega_k & \sigma_R & \cdots \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\omega_k^{2p-1} & \sigma_R^{2p-1}(y_k)_R & \omega_k^{2p-1} & \sigma_I^{2p-1}(y_k)_I & \cdots \\
\omega_k^{2p-2} & \sigma_R^{2p-2}(x_k)_R & \omega_k^{2p-2} & \sigma_I^{2p-2}(x_k)_I & \cdots \\
\omega_k^{2p-1} & \sigma_R^{2p-1}(x_k)_R & \omega_k^{2p-1} & \sigma_I^{2p-1}(x_k)_I & \cdots \\
& \vdots & & \vdots & \\
\omega_k & \sigma_R & \omega_k & \sigma_I & \cdots \\
\end{bmatrix}
\]

\[k = 1 \ldots N_f\]

The coefficients \(\sigma_R^o\) and \(\sigma_I^o\) are defined by,

\[
\sigma_R^o = \begin{bmatrix}
1, \ o = 4n \\
-1, \ o = 4n-1 \\
-1, \ o = 4n-2 \\
1, \ o = 4n-3
\end{bmatrix}, \quad \sigma_I^o = \begin{bmatrix}
1, \ o = 4n \\
1, \ o = 4n-1 \\
-1, \ o = 4n-2 \\
-1, \ o = 4n-3
\end{bmatrix}
\]

For response data corresponding to \(N_i\) initial conditions, or for frequency response functions corresponding to \(N_i\) reference locations, Equation (35) and (36) below can be derived similarly,
\[
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} & C_0 & \cdots & C_{2p} \\
\cdots & \omega_k^{2p-1} \sigma_R^{2p-1}(y_k)_R & \omega_k^{2p-1} \sigma_R^{2p-1}(y_k)_R & \cdots \\
\cdots & \omega_k^{2p-2} \sigma_R^{2p-2}(y_k)_R & \omega_k^{2p-2} \sigma_R^{2p-2}(y_k)_R & \cdots \\
\cdots & \vdots & \vdots & \cdots \\
\cdots & (y_k)_R & (y_k)_I & \cdots \\
\cdots & -\omega_k^{2p} \sigma_R^{2p} & 0 & \cdots \\
\cdots & 0 & -\omega_k^{2p-1} \sigma_R^{2p-1} & \cdots \\
\cdots & \vdots & \vdots & \cdots \\
\cdots & -I & 0 & \cdots \\
\end{bmatrix}
\]
\[
= [ \cdots -\omega_k^{2p} \sigma_R^{2p}(y_k)_R & -\omega_k^{2p} \sigma_I^{2p}(y_k)_I & \cdots ]
\]
\[
k = 1 \cdots N_f
\]

\[
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} & B_0 & B_1 & \cdots & B_{2p} \\
\cdots & \omega_k^{2p-1} \sigma_I^{2p-1}(H_k)_I & \omega_k^{2p-1} \sigma_I^{2p-1}(H_k)_R & \cdots \\
\cdots & \omega_k^{2p-2} \sigma_R^{2p-2}(H_k)_R & \omega_k^{2p-2} \sigma_I^{2p-2}(H_k)_I & \cdots \\
\cdots & \vdots & \vdots & \cdots \\
\cdots & (H_k)_R & (H_k)_I & \cdots \\
\cdots & -\omega_k^{2p} \sigma_R^{2p} & 0 & \cdots \\
\cdots & 0 & -\omega_k^{2p-1} \sigma_I^{2p-1} & \cdots \\
\cdots & \vdots & \vdots & \cdots \\
\cdots & -I & 0 & \cdots \\
\end{bmatrix}
\]
\[
= [ \cdots -\omega_k^{2p} \sigma_R^{2p}(H_k)_R & -\omega_k^{2p} \sigma_I^{2p}(H_k)_I & \cdots ]
\]
\[
k = 1 \cdots N_f
\]
Either of Equations (33), (35) or (36) can succinctly be written as,

$$X^t A^t \simeq B^t$$

For example, when Equation (37) stands for Equation (33), then $X^t$ is a matrix of dimension $(N_o, 2pN_o + 2pN_i + N_i)$ with the unknown parameters $A_1, \ldots, A_{2p}$ and $B_0, \ldots, B_{2p}$, $A^t$ is a coefficient matrix of dimension $(2pN_o + 2pN_i + N_i, 2N_f N_s)$ and $B^t$ a matrix of dimension $(N_o, 2N_f N_s)$ representing the right hand side. Equation (37) represents an overdetermined set of simultaneous linear equations when $2N_f N_s$ is larger than $(2pN_o + 2pN_i + N_i)$.

A more familiar form for Equation (37) is,

$$AX \simeq B$$

The least squares method has been used as pseudo-inverse procedure to estimate the unknowns in this equation.

Representing the pseudo-inverse of $A$ by $A^+$, the minimum norm least squares estimate for $X$, further on designated $\hat{X}$, equals [46, 47],

$$\hat{X} = A^+ B$$

The least squares estimate $\hat{X}$ has the property,

$$AX = B + E$$
and,

\[ (41) \quad \sum_{i=1}^{N_0} ||E_i||^2 = \sum_{i=1}^{N_0} ||A\hat{x} - E_i||^2 \text{ is minimum} \]

That is, \( \hat{x} \) minimizes the trace of \( E^tE \).

Rather than explicitly evaluating \( A^t \) and calculating \( \hat{x} \) from Equation (39), it is more efficient to calculate \( \hat{x} \) directly from the set of equations that can be derived from the explicit minimization of the criterion expressed by Equation (41), especially when \( A \) is full rank. Two such methods and their applicability to solve for the least squares estimates of the model parameters in Equation (33), (35) or (36), will be discussed in the next two sections.

2.2.2.2 Normal Equations Method

It is a classical result in least squares theory that the least squares solution \( \hat{x} \) of Equation (38) yields a set of residual vectors, represented by the columns of the matrix \( E \) in Equation (40), that are orthogonal to the vector subspace spanned by the columns of the coefficient matrix \( A \) [88]. Since \( A \) is real, this property is mathematically expressed by following equation,

\[ (42) \quad A^tE = 0 \]
Premultiplying both sides of Equation (40) by $A^t$ and using this latter property, it follows that $\hat{X}$ can be found as solution of following equation,

$$A^tA\hat{X} = A^tB$$

or,

$$P\hat{X} = D$$

The set of equations represented by Equation (44) are called the normal equations for the least squares problem defined by Equation (38). The coefficient matrix $P$, being the product of $A$ with it's transpose, is symmetric and positive semi-definite. Also, $P$ will be full rank when the matrix $A$ is full rank.

Any standard solution technique for a set of simultaneous linear equations can be used to solve Equation (44) for $\hat{X}$. Advantage can be taken of the fact that $P$ is symmetric and positive semi-definite and therefore can be decomposed in symmetric factors using a Cholesky decomposition ([89], pp. 142),

$$P = U^tU$$

Using the symmetric factorization of $P$, $\hat{X}$ can be solved from following two equations,

$$U^tY = D$$
(47) \( \hat{\mathbf{X}} = \mathbf{Y} \)

Equation (46) is solved for \( \mathbf{Y} \) by forward substitution, followed by solving Equation (47) for \( \hat{\mathbf{X}} \) by backward substitution.

The elements of the coefficient matrix \( \mathbf{P} \) are all weighted summations of the real or imaginary parts of the power spectra or cross power spectra of the measurement data at the different frequencies. The weighting coefficients are \( \omega_k \), raised to a certain power. For example, let \( A \) in Equation (38) correspond to the coefficient matrix of Equation (33). Denote the upper left partition of dimension \((N_o, N_o)\) in \( \mathbf{P} \) by \( P_{1,1} \). Then \( P_{1,1} \) equals,

(48) \[
P_{1,1} = \sum_{k=1}^{N_f} \omega_k^{p-2} (G_{Y_k Y_k})_R
\]

\( G_{Y_k Y_k} \) in Equation (48) is as defined by Equation (24).

The factors \( \omega_k \), raised to a power between 0 and 2p, in the coefficient matrix of Equation (33), (35) and (36), will cause the condition number of this matrix to be high. This will especially be the case, when data are used at frequencies covering a broad frequency range, and when \( p \), from Equation (2), is large. The condition number of \( \mathbf{P} \), being the product of \( \mathbf{A} \) with it's transpose, will be the square of the
condition number of $A$. The condition number has the interpretation of being an error magnification factor; it is my experience [76,77] that the normal equations for the least squares problem defined by Equation (33), (35) or (36) will frequently be ill-conditioned and cannot be solved accurately. The normal equations method is therefore not pursued in this work to estimate the parameters in the frequency domain models.

2.2.2.3 Method Using Orthogonal Decomposition of the Coefficient Matrix

Assume the coefficient matrix $A$ of the least squares problem defined by Equation (38) to be full rank. Let the dimension of $A$ in general be $(m,n)$. It can be proved that an orthogonal transformation exists, representable by a matrix $H$ of dimension $(m,m)$, with the property that ([88], pp. 11),

\begin{align}
(49) & \quad HA = R \\
(50) & \quad R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}
\end{align}

The matrix $R_1$ is of dimension $(n,n)$, upper triangular and full rank. The proof of the existence of such orthogonal transformation can be done by construction using Householder or Givens transformations.
Applying the orthogonal transformation defined by $H$ to both sides of Equation (40), it follows,

\begin{equation}
(51) \quad H\hat{x} = HB + HE
\end{equation}

or,

\begin{equation}
(52) \quad R\hat{x} = D + E'
\end{equation}

When $\hat{x}$ minimizes,

\begin{equation}
(53) \quad \sum_{i=1}^{N_0} ||\{E'\}_i||^2
\end{equation}

then $\hat{x}$ is the least squares solution of the problem defined by,

\begin{equation}
(54) \quad R\hat{x} = D
\end{equation}

But since $H$ represents an orthogonal transformation,

\begin{equation}
(55) \quad \sum_{i=1}^{N_0} ||\{E'\}_i||^2 = \sum_{i=1}^{N_0} ||\{E\}_i||^2
\end{equation}

Therefore, when $\hat{x}$ minimizes the left side of Equation (55), it also minimizes the right side of Equation (55). In other words, when $\hat{x}$ is the least squares solution of the problem defined by Equation (54), then $\hat{x}$ is also the least squares solution of the problem defined by Equation (38).

Equation (52) can however be solved directly for $\hat{x}$ that minimizes Equation (53). Let Equation (52) be partitioned conforming to the partitioning of $R$ in Equation (50),
\[
\begin{bmatrix}
R_1 \\
0
\end{bmatrix} \hat{X} = \begin{bmatrix}
D_1 \\
D_2
\end{bmatrix} + \begin{bmatrix}
E_1' \\
E_2'
\end{bmatrix}
\]

It follows,
\[
\sum_{i=1}^{N_0} \| \{E_i'\}_i' \|^2 = \sum_{i=1}^{N_0} (\| \{R_1 \hat{X} - D_1\}_i' \|^2 + \| \{D_2\}_i' \|^2)
\]

The minimum of this expression is achieved when,
\[
R_1 \hat{X} = D_1
\]

Equation (58) represents a set of simultaneous linear equations with a square upper triangular full rank coefficient matrix \( R_1 \), and can be solved for \( \hat{X} \) by backward substitution. The least squares error equals,
\[
\sum_{i=1}^{N_0} \| \{E_i'\}_i' \|^2 = \sum_{i=1}^{N_0} \| \{D_2\}_i' \|^2
\]

The importance of this method as solution technique for least squares problems, follows from the fact that the condition number of a matrix is not changed under orthogonal transformations. Hence the condition number of \( R \) defined in Equation (52) equals the condition number of \( A \). As was discussed in the previous section, the least squares problems defined by Equation (33), (35) or (36) may have a coefficient matrix, \( A \), with high condition number. Therefore it is important to use a solution method that yields an equation system, solvable for the least squares estimates,
in which the coefficient matrix has a condition number equal to, or of the same order as, the condition number of the coefficient matrix of the least squares problem. The method based on the orthogonal decomposition of $A$ has this property. The normal equations method does not have this property; it yields an equation system in which the condition number of the coefficient matrix equals the square of the condition number of $A$.

When the matrix $A$ is rank deficient, i.e. of rank $r$ and $r < \min(m,n)$, then two orthogonal transformations, representable by a matrix $H$ of dimension $(m,m)$ and a matrix $K$ of dimension $(n,n)$ respectively, exist with the property ([88], pp. 13),

\[(60) \quad HAK^t = R\]

\[(61) \quad R = \begin{bmatrix} R_1, 1 & 0 \\ 0 & 0 \end{bmatrix}\]

$R_1,1$ is an upper triangular matrix of dimension $(r,r)$. Define the following transformation of variables,

\[(62) \quad \hat{X} = \hat{Y}\]

After applying the orthogonal transformation defined by $H$ to both sides of Equation (40), and with the transformation of variables defined by Equation (62), it follows,

\[(63) \quad HAK^t\hat{X} = HB + HE\]
\{(64)\} \quad \hat{Y} = D + E' \\

When \( \hat{Y} \) minimizes,

\[
(65) \quad \sum_{i=1}^{N_0} ||\{E'\}_i||^2
\]

then \( \hat{Y} \) is a least squares solution of the least squares problem defined by,

\[
(66) \quad RY = D
\]

But since \( H \) represents an orthogonal transformation, it follows that \( \hat{X} \), related to \( \hat{Y} \) by Equation (62), is also the least squares solution of the problem defined by Equation (38).

Equation (64) can be partitioned, conforming to the partitioning of \( R \) in Equation (61),

\[
(67) \quad \begin{bmatrix} R_1, 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \begin{bmatrix} E'_1 \\ E'_2 \end{bmatrix}
\]

It follows,

\[
(68) \quad \sum_{i=1}^{N_0} ||\{E'\}_i||^2 = \sum_{i=1}^{N_0} (||\{R_1, 1 \hat{Y}_1 - D_1\}_i||^2 + ||\{\hat{Y}_2\}_i||^2)
\]

This expression is minimized when,

\[
(69) \quad R_1, 1 \hat{Y}_1 = D_1
\]

Equation (69) is a set of simultaneous linear equations with a square upper triangular full rank coefficient matrix \( R_1, 1 \),
and can be solved for \( \hat{\mathbf{y}}_1 \) by backward substitution.

The lower partition of \( \hat{\mathbf{y}}_1 \), \( \hat{\mathbf{y}}_2 \), can have an arbitrary value so that the least squares problem defined by Equation (67) has infinitely many solutions. The least squares solution that minimizes,

\[
\sum_{i=1}^{N_0} \| \{\hat{\mathbf{y}}_i\} \|^2
\]

is called the least squares solution of minimum length. The value of \( \hat{\mathbf{y}}_1 \) being determined by Equation (69), the least squares solution \( \hat{\mathbf{y}} \) that minimizes Equation (70) is obtained when \( \hat{\mathbf{y}}_2 \) equals 0. Since \( K \) represents an orthogonal transformation, \( \hat{\mathbf{x}} \) calculated from Equation (62) will be the least squares solution of minimum length for the problem defined by Equation (38) when \( \hat{\mathbf{y}} \) is the least squares solution of minimum length for the problem defined by Equation (66).

The coefficient matrix in Equation (33), (35) and (36) will be full rank under the conditions specified in Section 2.2.1. An orthogonal decomposition of the coefficient matrix defined by Equation (49), followed by the solution of Equation (58), has been found to be an acceptable method to find the least squares estimates for the parameters in the model defined by Equation (2.1.38), (2.1.43) or (2.1.48).
2.2.3 Use of Principal Component Response Data

This section is primarily concerned with the case of a system with $N$ effective degrees of freedom, whose response is observed at $N_c$ response locations and $N_c$ is considerably larger than $N$. This case is of great practical importance in experimental modal analysis; the response is generally observed over a frequency range with a limited number of modes, but at many response locations, to obtain an accurate description of the mode shapes.

The models defined by Equation (2.1.38), (2.1.43) and (2.1.48), with $p$ equal to 1, become large order incomplete models for this situation, describing the data in terms of many more modes than actually are observable. The conditions expressed by Equation (5), (26) or (28) are not fulfilled, making the model parameters not uniquely defined by the data. This in itself is not a reason for not using such a large order incomplete model, since it is always possible to obtain a set of parameters, for example the set of parameters with minimum norm. But the procedure becomes more complicated and the computational cost higher, especially for the spectral analysis of the model to obtain the modal parameters.
When $N_0$ is larger than $N$, then the $N_0$ response locations do not any longer define a set of generalized coordinates. Let the response at a set of $N$ generalized coordinates be represented by $\{y_k^i\}_i$, a vector of $N$ elements. Then the response at the $N_0$ response locations can be related to the response at the generalized coordinates by,

\[(71) \quad \{y_k\}_i = C\{y_k^i\}_i \quad k = 1 \ldots N_f, \quad i = 1 \ldots N_g\]

The matrix $C$ is a matrix of dimension $(N_0, N)$, that can be assumed real and with orthogonal columns since $\{y_k^i\}_i$ represents the response of a mechanical system. Indeed, $N_0$ being larger than $N$, let $\{y_k\}_i$ be related to $\{x_k\}_i$ by the model defined in Equation (3) with $p$ equal to 1,

\[(72) \quad A^2(s_k)\{y_k\}_i = B^2(s_k)\{x_k\}_i \quad s_k = j\omega_k\]

\[A^2(s) = Is^2 + A_1s + A_2\]

\[B^2(s) = B_0s^2 + B_1s + B_2\]

Substituting $\{y_k\}_i$ from Equation (71) in Equation (72), and premultiplying both sides of Equation (72) with $C^t$, it follows,

\[(73) \quad A'^2(s_k)\{y_k^i\}_i = B'^2(s_k)\{x_k\}_i \quad s_k = j\omega_k\]

\[A'^2(s) = C^tA^2(s)C\]

\[B'^2(s) = C^tA^2(s)C\]

The coefficients of the polynomials $A'^2(s)$ and $B'^2(s)$ in
Equation (73) remain real. Since the columns of \( C \) are orthogonal, the highest order coefficient of \( A^2(s) \) remains equal to \( 1 \). \( \{Y_k\}_i \) is therefore related to \( \{X_k\}_i \) by a parametric model as defined by Equation (3) with \( p \) equal to \( 1 \). This parametric model describes a system with exactly \( N \) modes.

The condition expressed by Equation (5) can not be satisfied when \( N_0 \) is larger than \( N \); substituting \( \{Y_k\}_i \) from Equation (71) in the expression for \( (G_{YY})_R \), Equation (22), it is directly verified that the rank of \( (G_{YY})_R \) equals \( N \) rather than \( N_0 \). Then the coefficients in the matrix polynomials \( A^2(s) \) and \( B^2(s) \) of the model by Equation (72) are not unique, and the coefficient matrix of the corresponding least squares problem will be rank deficient. As already mentioned, it remains possible to solve the least squares equation for the minimum length solution, but it complicates the solution procedure. Also, the size of the eigenvalue decomposition, that needs to be solved for the pole values and mode shapes, will be of order \( 2N_0 \), while it preferably should be of order \( 2N \) or somewhat larger depending on how noisy the data is. Especially when \( N_0 \) is much larger than \( N \), this is an important consideration; not only will the eigenvalue decomposition become prohibitively expensive, but it also may become difficult to distinguish the \( N \) system modes among the \( N_0 \) calculated modes.
A procedure will be presented to determine the matrix \( C \) and corresponding response vectors \( \{ Y_k^i \}_i \) at \( N \) generalized degrees of freedom from the response data \( \{ Y_k \}_i \) at the \( N_0 \) response locations. Since the effective number of modes is actually unknown, the procedure also generates an indication of \( N \).

Let \( \{ Y_k^i \}_i \) represent the response vector at \( N_0 \) response locations and related to \( \{ Y_k \}_i \) by a real matrix \( B \) of dimension \((N_0^i, N_0)\),

\[
(74) \quad \{ Y_k^i \}_i = B \{ Y_k \}_i \\
\quad \quad \quad k = 1 \ldots N_f , \quad i = 1 \ldots N_s
\]

Let \( \{ \hat{Y}_k \}_i \) be related to \( \{ Y_k^i \}_i \) by a real matrix \( C \) of dimension \((N_0, N_0^i)\),

\[
(75) \quad \{ \hat{Y}_k \}_i = C \{ Y_k^i \}_i
\]

\( \{ \hat{Y}_k \}_i \) can be viewed as an approximation of \( \{ Y_k \}_i \). Let \( \{ E_k \}_i \) represent the difference between \( \{ Y_k \}_i \) and \( \{ \hat{Y}_k \}_i \),

\[
(76) \quad \{ E_k \}_i = \{ Y_k \}_i - \{ \hat{Y}_k \}_i = \{ Y_k \}_i - CB \{ Y_k \}_i
\]

Since both matrices \( B \) and \( C \) are real, Equation (76) can be split in real and imaginary part,

\[
(77) \quad \{ E_k \}_i R = [I - CB] \{ Y_k \}_i R
\]
(78) \((\{E_k\}_i)_I = [I - CB](\{Y_k\}_i)_I\)

All error vectors can then be represented by one real error matrix,

(79) \(E = [ \ldots (\{E_k\}_i)_R (\{E_k\}_i)_I \ldots ]\)

The idea is to find matrices \(B\) and \(C\) so that \(\{\hat{Y}_k\}_i\) approaches \(\{Y_k\}_i\) as well as possible. Such matrices are found as the matrices \(B\) and \(C\) that minimize simultaneously all singular values of,

(80) \(EE^t\)

and can be proved to be,

\[
\begin{bmatrix}
\{U\}^t_1 \\
\vdots \\
\{U\}^t_{N_0}
\end{bmatrix}
\]

(81) \(B = \begin{bmatrix}
\{U\}^t_1 \\
\vdots \\
\{U\}^t_{N_0}
\end{bmatrix}\)

(82) \(C = B^t\)

\(\{U\}_i, i = 1, \ldots, N_0\), are the singular vectors corresponding to the \(N_0\) largest singular values of \((G_{YY})_R\), defined by Equation (22). Since \((G_{YY})_R\) is real, symmetric and positive semi-definite, the singular value decomposition of \((G_{YY})_R\) equals,

(83) \((G_{YY})_R = USU^t\)

\(S\) is a diagonal matrix of dimension \((N_0, N_0)\) with the
singular values $s_j$ of $(G_{YY})_R$, in descending order on the diagonal. $U$ stands for the matrix of dimension $(N_o, N_o)$ with the singular vectors, and is orthogonal. The error matrix $E E^t$ corresponding to this choice of $B$ and $C$ equals,

$$EE^t = \sum_{j=N_o+1}^{N_o} s_j [U]_j [U]^t$$

Note that when $N_o'$ equals $N_o$, then the above matrix equals 0 and $[\hat{Y}_k]_i$ equals $[Y_k]_i$ for all $k$ and $i$.

The results indicated by above equations are some basic results of the theory of principal component analysis of vector valued variables ([91], Chapter 8, [92], pp. 435-440, [93], pp. 125-127). The response data represented by $[Y_k']_i$ will be called the principal component response data corresponding to the original response data $[Y_k]_i$.

Let $(G_{Y', Y'})_R$ be the matrix defined by,

$$\sum_{k=1}^{N_f} (G_{Y'k} Y'_k)_R$$

The principal component response data can be proved to have the following property ([91], pp. 340),

$$S' = S$$

$S'$ is a diagonal matrix of dimension $(N'_o, N'_o)$ with the $N'_o$ largest singular values of $(G_{YY})_R$ in descending order on the diagonal. The proof of this property follows from the definition of $[Y'_k]_i$, by Equation (74), with $B$ defined by
Equation (81). The principal component response data therefore consists of linear combinations of the original response data that are uncorrelated.

Finally let \((G_{YY}^{\hat{}})_R\) be the matrix of dimension \((N_O,N_O)\) defined by,

\[
(87) \quad (G_{YY}^{\hat{}})_R = \sum_{i=k}^{N_f} (G_{Y_kY_k}^{\hat{}})_R
\]

Then it can be proved that,

\[
(88) \quad ||(G_{YY})_R - (G_{YY}^{\hat{}})_R||^2 = \sum_{j=N_0+1}^{N_O} \varepsilon_j^2
\]

Recall that the response data represented by \(\{\hat{Y}_k\}_i\), derived from the principal component response data \(\{Y'_k\}_i\) by Equation (75), is an approximation for the actual response data \(\{Y_k\}_i\). The above property then indicates that the real part of the power spectrum matrices of \(\{\hat{Y}_k\}_i\), summed over the \(N_f\) frequencies, approaches the real part of the power spectrum matrices of \(\{Y_k\}_i\) summed over the \(N_f\) frequencies.

Assume now that \(\{Y_k\}_i\) represents exact theoretical noise free response data at \(N_O\) locations of an \(N\) degree of freedom system. Then exactly \(N\) singular values of \((G_{YY})_R\) will be non-zero. To prove this, suppose that the number of non-zero singular values does not equal \(N\), but for example is smaller. Select \(N'_0\) equal to the number of non-zero singular values. \(\{\hat{Y}_k\}_i\), defined by Equation (76), equals exactly
\( \{Y_k\}_i \), since the error matrix defined by Equation (80) becomes 0. The principal component response \( \{Y_k'\}_i \) is described by a model, defined by Equation (73), that only describes a system with \( N_0' \) modes. But then so is \( \{Y_k\}_i \), related to \( \{Y_k'\}_i \) by Equation (76), which is in contradiction with the supposition that \( \{Y_k\}_i \) describes the response of a system with \( N \) modes. It can similarly be proved that the number of non-zero singular values cannot be larger than \( N \).

Practically, the response data \( \{Y_k\}_i \) will not be noise free, and therefore none of the singular values will actually equal 0. The series of singular values, in descending order, might however be broken after \( N_0' \) values, so that the remainder singular values can be considered negligible small and therefore the error matrix defined in Equation (80). Numerically, \( N_0' \) can be considered as the pseudo-rank of \( (G_{YY})_R \). It may be expected that \( N_0' \), characterizing such breakpoint, will be somewhat larger than \( N \), but, especially when \( N_0 \) is much larger than \( N \), still considerable smaller than \( N_0 \).

The principal component response data \( \{Y_k'\}_i \), defined by Equation (74), can now be used as data together with \( \{X_k\}_i \) to estimate a parametric model of the form defined by Equation (3) that is low order complete. The order \( p \) of this model, and therefore the number of modes \( N_m \) described by this model, is selected from.
(89) \[ N_m = pN'_o \quad , \quad N_m = N + N_c \]

Since \( N'_o \) approximates \( N \), the order \( p \) will generally be small, in the range from 1 to 4 depending on how noisy the data is. The matrix \((G_{Y',Y'})_R\) is full rank, and the model parameters are uniquely described by the data.

The parameters of the model being estimated, they can be used to calculate an estimate for the upper residuals, for the pole values and the corresponding mode shapes by an eigenvalue decomposition of size \( 2N_m \), and for the modal participation factors, using Equation (2.1.58), (2.1.59) and (2.1.66). Let the estimate of the \( 2N_m \) pole values be represented by a diagonal matrix \( \hat{\Lambda} \) of dimension \( (2N_m,2N_m) \) and of the corresponding mode shapes by a matrix \( \hat{V}' \) of dimension \( (N'_o,2N_m) \). Also let the estimate of the modal participation factors be represented by a matrix \( \hat{\Gamma} \) of dimension \( (2N_m,N'_i) \), and of the upper residuals by a matrix \( \hat{U}' \) of dimension \( (N'_o,N'_i) \). Then a canonical response expression for \( \{X'_k\}_i \) is of the form,

(90) \[ \{Y'_k\}_i \approx \hat{V}'[sI - \hat{\Lambda}]^{-1}\hat{\Gamma}\{X'_k\}_i + \hat{U}'\{X'_k\}_i \]

The approximation sign \( \approx \) is used since the parameters in the canonical response expression are estimates. Using Equation (75) a canonical response expression for \( \{\hat{Y}'_k\}_i \) is of the form,
\[
\{ \hat{y}_k \}_i = \mathbf{C} \mathbf{V}' [s\mathbf{I} - \hat{\mathbf{A}}]^{-1} \hat{\mathbf{L}} \{ x_k \}_i + \hat{\mathbf{U}}' \{ x_k \}_i
\]

\[
\{ \hat{x}_k \}_i = \hat{\mathbf{V}} [s\mathbf{I} - \hat{\mathbf{A}}]^{-1} \hat{\mathbf{L}} \{ x_k \}_i + \hat{\mathbf{U}} \{ x_k \}_i
\]

But since \( N_o \) is selected such that the error function defined by Equation (84) is small, \( \{ \hat{y}_k \}_i \) approximates \( \{ y_k \}_i \) and it follows that,

\[
\{ y_k \}_i \approx \hat{\mathbf{V}} [s\mathbf{I} - \hat{\mathbf{A}}]^{-1} \hat{\mathbf{L}} \{ x_k \}_i + \hat{\mathbf{U}} \{ x_k \}_i
\]

Therefore, \( \hat{\mathbf{A}}, \hat{\mathbf{V}}, \hat{\mathbf{L}} \) and \( \hat{\mathbf{U}} \), defined in Equation (92), represent an approximation of the modal parameters used in the canonical response expression for \( \{ y_k \}_i \) using \( N_m \) modes, \( N \) of which are structure modes and \( N_o \) modes to describe the noise.

When the data are available as frequency response functions at \( N_o \) response locations for \( N_i \) reference locations, and \( N_o \) is larger than \( N \), then it is again advantageous to derive principal component frequency response functions for estimating a low order parametric model. The appropriate equations can be summarized as follows,

\[
G_{HH} = \sum_{k=1}^{N_f} H_k H_k^h
\]

\[
(G_{HH})_R = U S U^t
\]

\[
H_k' = B H_k
\]

\[
\hat{H}_k' = C H_k
\]
B and C in Equation (96) and (97) are as defined by Equation (81) and (82) respectively, with U from Equation (95).

Finally when the data are free decay response data for \( N_0 \) response locations and \( N_1 \) initial conditions, \( N_0 \) again larger than \( N \), then principal component free decay response data, to which the parametric model can be estimated, are defined by,

\[
G_{YY} = \sum_{k=1}^{N_f} Y_k Y_k^h
\]

(98) \( (G_{YY})_R = USU^t \)

(99) \( Y_k' = BY_k \)

(100) \( \hat{Y}_k = CY_k' \)

(101)
2.2.4 Some Numerical and Practical Considerations

As already mentioned in Section 2.2.2, the condition number of the matrix $A$ in the least squares problem defined by Equation (38) will generally be high when $A$ stands for the coefficient matrix in either Equation (33), (35) or (36). Therefore the normal equations, summarized by Equation (44) and in which the coefficient matrix has a condition number equal to the square of the condition number of $A$, can frequently not be solved accurately. The method based on the orthogonal composition of $A$, summarized by Equation (49), (50) and (58) when $A$ is full rank, is recommended.

Let $A$ be a full rank matrix of dimension $(m,n)$, and let $H$ be a matrix of dimension $(m,m)$ that represents an orthogonal transformation with the property expressed by Equation (49) and (50), i.e.,

$$\text{(102)} \quad R = HA$$

A lower bound for the condition number of $A$, represented by $\kappa(A)$, can be proved to be ([88], pp. 29),

$$\text{(103)} \quad \kappa(A) \geq \frac{\max(|r_{i,j}|)}{\min(|r_{i,i}|)} \quad i < j \quad \text{and always} \quad \kappa(A) \geq 1$$

In the last equation, $r_{i,j}$ represents an element of $R$. The lower bound expressed by Equation (103) is not a conserva-
tive lower bound, in some cases known to be considerably smaller than the actual value of Κ(A). It can be used however to indicate the influence of the frequency range and the order of the direct parameter model on the condition number.

For example, let A stand for the transpose of the coefficient matrix in Equation (35), and let \( N_f \) frequencies cover a base band with maximum frequency \( \omega_{\text{max}} \). Since H represents an orthogonal transformation, it follows that \( \text{max}(|r_{i,j}|) \) will be larger than \( \omega_{\text{max}}^{2p} \) and that \( \text{min}(|r_{i,i}|) \) will be smaller than \( N_f \). An even less conservative lower bound for the condition number of this matrix therefore is,

\[
(104) \quad \text{Κ}(A) > \frac{\omega_{\text{max}}^{2p}}{N_f}, \text{ and always } \text{Κ}(A) \geq 1
\]

<table>
<thead>
<tr>
<th>( \omega_{\text{max}} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>4E1</td>
<td>4E2</td>
</tr>
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<td>4E1</td>
<td>4E3</td>
<td>4E5</td>
</tr>
<tr>
<td>1000</td>
<td>1</td>
<td>4E3</td>
<td>4E6</td>
<td>4E9</td>
</tr>
</tbody>
</table>

\( \omega \) (rad/sec)

**Table 2.1** Lower Bounds for Condition Number, Base Band 0 - \( \omega_{\text{max}} \)
Table 2.1 summarizes some lower bounds of the condition numbers calculated with Equation (104) for several values of $\omega_{\max}$ and $p$, assuming $N_f$ equals 256. The deterioration of the condition number with increasing value of $\omega_{\max}$ and $p$ is obvious.

As indicated in Section 2.2.3, in many situations of practical interest in modal analysis, $N_o$ will be considerable larger than $N$. After determining $N_o$ from the rank study of $(G_{YR})_R$ or $(C_{HH})_R$, whichever applies, and calculating the corresponding principal component response data, a model as defined by Equation (3) with order $p$ limited to the range from 1 to 4 depending on the noise level, will suffice to describe the principal component response data. Thus, the condition number of $A$ can be kept from growing very large, especially when the frequency range is not broad.

When $N_o$ is a lot smaller than $N$, then the order $p$ of the model defined by Equation (3) may be considerably larger to meet the requirements set forth by Equation (1) and (2), causing the condition number of $A$ to deteriorate. The least squares problem defined by Equation (38) cannot be solved accurately for the model parameters in such cases.

The condition number of the matrix $A$ can be improved by a proper scaling of the frequency values. In particular, the condition number can be lowered by a scaling of the frequen-
cy values between 0 and 1, an experience shared with others [27]. This is also immediately obvious from Table 2.1. The actual decrease of the condition number may however not be as significant as the dramatic decrease of the tabulated lower bounds of the condition number, defined by Equation (104).

The parametric model that is identified on the data will primarily be used to calculate modal parameters, i.e. natural frequencies, modal damping values, mode shapes and modal participation factors. When indication exists of frequency ranges in which the natural frequencies of the system are located, as for example when the data consists of frequency response functions, then one might consider weighting the data in such ranges more than the data at frequencies in between natural frequencies. In particular the estimates of the modal damping values and also, to a lesser extent, of the residues have been found to be influenced by such weighting.

Consider the least squares problem defined by Equation (38) with a coefficient matrix A of dimension \((m,n)\). The weighting of the data at different frequencies can be represented by a diagonal matrix \(W\) of dimension \((m,m)\) that premultiplies both sides of Equation (38),

\[
(105) \quad WAX = WB
\]
For example, the diagonal entries in $\mathbf{W}$ may be high for rows in $\mathbf{A}$ that contain data at frequencies around the resonance frequencies and low for rows that contain data at frequencies in between.

The least squares solution of Equation (38) minimizes Equation (41), the least squares solution $\hat{\mathbf{x}}$ of Equation (105) minimizes,

$$\sum_{i=1}^{N_0} ||\{WE\}_i||^2 = \sum_{i=1}^{N_0} ||\{W\mathbf{A}\mathbf{x} - WB\}_i||^2$$

Figure 2.1 shows an example of weighting: the values of a frequency response function at frequencies in bands of 100 Hz centered around the resonance peaks are weighted ten times more than the values at frequencies in between resonance frequencies.

The effect of using different weighting schemes, is reflected mostly in changes of the modal damping values and residues. When the data around resonance frequencies are weighted higher than data in between resonance frequencies, then this will yield lower values of modal damping. Weighting the data in between resonance frequencies higher than the data around resonance frequencies will have an opposite effect. The changes are however not significant nor systematic enough to justify the additional overhead for including weighting.
schemes. Also the selection of the weighting scheme would primarily depend on operator judgement and would add an additional factor of arbitrariness to the estimation procedure.
Figure 2.1 Weighted Least Squares
2.3 Direct Parameter Model Identification in the Time Domain

Estimation procedures for the parameters in the models for sampled time data will be discussed in this section. The parametric models were derived in Section 2.1.3, and are, depending on data type, summarized by Equation (2.1.75), (2.1.87) and (2.1.88) in that section.

The development in this section parallels substantially the development of the estimation procedures for frequency domain models. This is not surprising, since both models are merely representations in two different domains of the same differential equation, defined in Section 2.1.1 by Equation (2.1.18).

It will again be assumed that the noise on the data can be accounted for by including complementary degrees of freedom, equivalent to $N_c$ modes in the parametric model. The total number of modes described by the parametric model, $N_m'$ and the order of the model, $p$, are related to $N$, $N_c$ and $N_o$ by Equation (2.2.1) and (2.2.2). Also, various logical data sets of sampled data $\{x_k\}$ and $\{y_k\}$ may be available. To
distinguish such data sets, an additional index is introduced and the notation for \( \{x_k\} \) and \( \{y_k\} \) is changed to \( \{x_k\}_i \) and \( \{y_k\}_i \). Assuming \( N_S \) data sets, all columns \( \{x_k\}_i \) and \( \{y_k\}_i \), \( i = 1 \ldots N_S \), can be represented by the matrices \([x_k]\) and \([y_k]\) of dimension \((N_i,N_S)\) and \((N_o,N_S)\) respectively. For example, \( \{x_k\}_i \), \( i = 1 \ldots N_S \), may represent \( N_S \) sets of sampled transient force input signals, and \( \{y_k\}_i \) the corresponding responses. The total number of time samples in every data set will be assumed equal to \((N_t+1)\).

The equations that will be derived for time domain direct parameter model identification will be shown to have some very interesting characteristics that make them a lot more appealing and applicable than the corresponding equations derived for frequency domain direct parameter model identification. For example, the condition number of the coefficient matrix in the equation system to be solved for the model parameters is generally low and independent of the order of the parametric model. The normal equations method will produce a set of well conditioned equations and therefore does present a viable solution method for the least squares problem. Moreover, for certain kinds of data, the coefficient matrix of the normal equations will have the structure of a block Toeplitz matrix, making it possible to solve the equations by a fast recursive algorithm, such as the Levinson-Wiggins-Robinson (L.W.R.) algorithm.

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Sometimes, it is also possible to split the estimation procedure in two phases. In the first phase, the parameters of the homogeneous part, or the autoregressive part, of the model are estimated, and if required the parameters of the moving-average part in a second phase. The first phase will then yield all necessary information to calculate the characteristic solutions, i.e. pole values and mode shapes, which frequently may be all that is required.

2.3.1 Conditions for Uniqueness of the Parameters

As for the frequency domain direct parameter model identification, some conditions can be developed that warrant the uniqueness of the parameters in the time domain models. These conditions will help determine when the equations to be solved for the parameters are full rank.

Consider Equation (2.1.87), succinctly written as,

\[ A^{2p}(q^{-1})y_k = B^{2p-1}(q^{-1})x_k \]

\[ i = 1 \ldots N_S \]

\[ A^{2p}(q^{-1}) = I - A_1 q^{-1} - \ldots - A_{2p} q^{-2p} \]

\[ B^{2p-1}(q^{-1}) = B_0 + B_1 q^{-1} + \ldots + B_{2p-1} q^{-2p+1} \]

The range of the index \( k \), indicating the time samples to be
used, will further be discussed in Section 2.3.2. Let it be such that in every data set \((N_t+1)\) time samples are available. For the coefficients of the polynomials \(A^{2p}(q^{-1})\) and \(B^{2p-1}(q^{-1})\) to be uniquely defined by the data \(\{x_k\}_i\) and \(\{y_k\}_i, k = 0 \ldots N_t\) and \(i = 1 \ldots N_s\), the following two conditions need to be fulfilled,

(2) \([x]\) has rank \(N_i\)
(3) \([y]\) has rank \(N_o\)

where \([x]\) and \([y]\) stand for,

(4) \([x] = [\{x_0\}_1 \ldots \{x_{N_t}\}_N_s]\)
(5) \([y] = [\{y_0\}_1 \ldots \{y_{N_t}\}_N_s]\)

The conditions expressed by Equation (2) and (3) can be proved to be necessary and sufficient, using directly the conditions expressed by Equation (2.2.4) and (2.2.5), that warrant the uniqueness of the frequency domain direct parameter model. Let hereto \(\{X_k\}_i\) and \(\{Y_k\}_i, k = 1 \ldots N_r\), represent values of the discrete Fourier transforms of \(\{x_k\}_i\) and \(\{y_k\}_i, k = 0 \ldots N_t\). Assume that the conditions expressed by Equation (2.2.4) and (2.2.5) are not satisfied. Then invertible real matrices \(S_i\) and \(S_o\) exist such that,

(6) \(S_i[(X)_R (X)_I] = [(X)_R (X)_I]\)
(7) \(S_o[(Y)_R (Y)_I] = [(Y)_R (Y)_I]\)
or, written in complex form,

\[(8) \quad S_i X = X\]
\[(9) \quad S_o Y = Y\]

Taking the discrete Fourier transform of the \(N_s\) data sets represented by \(X\) and by \(Y\), and using the linearity of the transform, Equation (8) and (9) transform to,

\[(10) \quad S_i [x] = [x]\]
\[(11) \quad S_o [y] = [y]\]

Equation (10) and (11) indicate that the matrices \([x]\) and \([y]\) are rank deficient, so that the conditions expressed by Equation (2) and (3) are not fulfilled. Since no unique frequency domain direct parameter model exists for the given data, no unique time domain direct parameter model can exist for the corresponding time domain data. The latter is therefore implied when the conditions expressed by Equation (2) and (3) are not satisfied.

Without making use of transform relationships, the conditions expressed by Equation (2) and (3) could also be proved to be necessary and sufficient using arguments similar to the ones used in Section 2.2.1.

Alternatively, above conditions can be stated as,

\[(12) \quad R_{xx}(0) \text{ has rank } N_i\]
(13) $R_{yy}(0)$ has rank $N_o$

where,

(14) $R_{xx}(0) = \sum_{k=0}^{N_t} \left( \sum_{i=1}^{N_s} \{x_k\}_i \{x_k\}_i^t \right) = [x][x]^t$

(15) $R_{yy}(0) = \sum_{k=0}^{N_t} \left( \sum_{i=1}^{N_s} \{y_k\}_i \{y_k\}_i^t \right) = [y][y]^t$

To satisfy both conditions, it is at least required that,

(16) $N_s(N_t+1) \geq \max(N_i, N_o)$

As already indicated in Section 2.2.1, the force input signals will generally be such that the condition expressed by Equation (2) is satisfied. This is not so for the conditions on the output signals, expressed by Equation (3); especially when $N_o$ is considerably larger than $N$, the latter condition may not be fulfilled.

Similarly, the uniqueness of the parameters in the model for the sampled impulse response function matrix $H_k$, defined by Equation (2.1.75), is governed by the condition that,

(17) $H = [H_0 \ldots H_{N_t}]$ has rank $N_o$

or,

(18) $R_{HH}(0) = \sum_{k=0}^{N_t} H_k H_k^t$ has rank $N_o$
Finally, the parameters in the model for the free decay response, defined by Equation (2.1.88), will be unique when,

\[(19) \quad [y] = [[y_1] \ldots [y_N]] \text{ has rank } N_0\]

or,

\[(20) \quad R_{yy}(0) = \sum_{k=0}^{N_t} [y_k][y_k]^t \text{ has rank } N_0\]

Assuming impulse response functions for \(N_i\) reference locations, or free decay responses for \(N_i\) initial conditions, above conditions require at least that,

\[(21) \quad N_i(N_t+1) \geq N_0\]

2.3.2 Least Squares Estimates

2.3.2.1 Least Squares Problem Formulation

Consider the time domain model defined in Section 2.1.3 by Equation (2.1.87), compactly represented by Equation (1). Since the zero lag coefficient in \(A^2P(q^{-1})\) equals I, the factors in this equation can be reorganized to yield,
(22) \( \{y_k\}_i = A_1\{y_{k-1}\}_i + \cdots + A_{2p}\{y_{k-2p}\}_i \\
+ B_0\{x_k\}_i + B_1\{x_{k-1}\}_i + \cdots + B_{2p-1}\{x_{k-2p+1}\}_i \)
\[ i = 1 \ldots N_s \]

Physically, Equation (22) can be interpreted as a prediction equation for \( \{y_k\}_i \) by its own past values \( \{y_{k-j}\}_i \), \( j = 1 \ldots 2p \), weighted with the coefficients \( a_j \), and by the values of the input \( \{x_{k-j}\}_i \), \( j = 0 \ldots (2p-1) \), weighted with the coefficients \( B_j \). Indicating the prediction error, or residual, by \( \{e_k\}_i \), Equation (22) can more precisely be written as,

(23) \( \{y_k\}_i = A_1\{y_{k-1}\}_i + \cdots + A_{2p}\{y_{k-2p}\}_i \\
+ B_0\{x_k\}_i + B_1\{x_{k-1}\}_i + \cdots + B_{2p-1}\{x_{k-2p+1}\}_i + \{e_k\}_i \)
\[ i = 1 \ldots N_s \]

Assuming the structure to be time invariant, then Equation (23) can be used as prediction equation for a number of values of the index \( k \). The range of \( k \) is however deliberately not indicated yet. Let, in every data set, \( (N_t+1) \) time samples be available, i.e. \( \{x_j\}_i \) and \( \{y_j\}_i \), \( j = 0 \ldots N_t \). To specify the range for the index \( k \), one needs to distinguish the following cases [94] :

1. **Prewindowed case.** The values for \( \{x_j\}_i \) and \( \{y_j\}_i \) equal 0 for \( j < 0 \), i.e. before time zero. Then the range for \( k \) can be taken equal to,
(24) \( k = 0 \ldots N_t \)

The prewindowed case can reasonably be assumed when \( \{y_j\}_i \) represents the response of a causal system initially at rest to a force input applied from time zero on.

2. **Pre- and postwindowed case.** The values for \( \{x_j\}_i \) and \( \{y_j\}_i \) equal zero for \( j < 0 \), i.e. before time zero, and for \( j > N_t \), i.e. after the observation period \( T \), of length \( N_t \Delta t \). The range for \( k \) can be choosen to be,

(25) \( k = 0 \ldots (N_t+2p) \)

The pre- and postwindowed case can reasonably be assumed when \( \{y_j\}_i \) represents the transient response of a causal system to a transient force input \( \{x_j\}_i \) applied from time zero on, and both \( \{x_j\}_i \) and \( \{y_j\}_i \) have died out in the observation period. The pre- and postwindowed case is also known as the **correlation case** (from the special structure of the normal equations, as will be explained further on).

3. **Non-windowed case.** The values of input \( \{x_j\}_i \) and response \( \{y_j\}_i \) are essentially unknown outside the observation period. The range for \( k \) then equals,

(26) \( k = 2p \ldots N_t \)

The non-windowed case is appropriate, for example when \( \{y_j\}_i \) represents the response to a stationary random force input
\{x_j\}_i$. The non-windowed case is also known as the **covariance case**.

The covariance case can be considered as the most general case. Therefore the range for \( k \) will in general be indicated as \( k = N_b \ldots N_e \). For the covariance case, specified above, \( N_b \) equals \( 2p \) and \( N_e \) equals \( N_t \). For the pre- and postwindowed case \( N_b \) equals \( 0 \) and \( N_e \) equals \( (N_t + 2p) \), and for the prewindowed case \( N_b \) equals \( 0 \) and \( N_e \) equals \( N_t \). The different window cases are further illustrated in Figure 2.2.

![Figure 2.2 Various Window Cases for Time Domain Data Analysis](image)
Let all columns \( \{e_k\}_i \) in Equation (23), for \( i = 1 \ldots N_s \), be represented by \([e_k]\). All equations like Equation (23), for \( k = N_b \ldots N_e \), can be organized in one matrix equation,

\[
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} & B_0 & B_1 & \cdots & B_{2p-1} \\
[y_{N_b-1}] & \cdots & [y_{N_e-1}] \\
[y_{N_b-2}] & \cdots & [y_{N_e-2}] \\
\vdots & & \vdots \\
[y_{N_b-2p}] & \cdots & [y_{N_e-2p}] \\
[x_{N_b}] & \cdots & [x_{N_e}] \\
[x_{N_b-1}] & \cdots & [x_{N_e-1}] \\
\vdots & & \vdots \\
[x_{N_b-2p+1}] & \cdots & [x_{N_e-2p+1}] \\
\end{bmatrix}
\]

(27)

\[
= [[y_{N_b}] \ldots [y_{N_e}]] - [[e_{N_b}] \ldots [e_{N_e}]]
\]

Or compactly,

(28) \( X^t A^t = B^t + E^t \)

(29) \( AX = B + E \)

(30) \( AX = B \)

Equation (30) represents an overdetermined set of equations when \( N_s(N_e-N_b+1) \) is larger than \( 2p(N_c+N_1) \). As for the frequency domain models, the least squares method will be used to solve this overdetermined equation system.
It is sometimes convenient to write Equation (23) alternatively as,

\[(31) \{y_k\}_i = C_0\{z_k\}_i + \cdots + C_{2p-1}\{z_{k-2p+1}\}_i + \{e_k\}_i \]
\[k = N_b \ldots N_e, \quad i = 1 \ldots N_s\]

The matrices \(C_i\) are of dimension \((N_0, N_0+N_i)\), and the vectors \(\{z_j\}_i\) have \((N_0+N_i)\) elements. \(C_i\) and \(\{z_j\}_i\) are defined by,

\[(32) C_j = [A_j B_j], \quad \{z_j\}_i = \begin{bmatrix} \{y_{j-1}\}_i \\ \{x_j\}_i \end{bmatrix}\]

Equation (31) is in the form of a multichannel filter of order \(2p\) with \((N_0+N_i)\) inputs, represented by \(\{z_j\}_i\), and \(N_0\) outputs, represented by \(\{y_j\}_i\). All equations like Equation (31) can be written in the form,

\[
\begin{bmatrix}
C_0 & C_1 & \cdots & C_{2p-1}
\end{bmatrix}
\begin{bmatrix}
[z_{N_b}] & [z_{N_b+1}] & \cdots & [z_{N_e}]
\end{bmatrix}
\]

\[(33) = \begin{bmatrix}
[z_{N_b}] & [z_{N_b+1}] & \cdots & [z_{N_e}]
\end{bmatrix} - \begin{bmatrix}
[e_{N_b}] & [e_{N_b+1}] & \cdots & [e_{N_e}]
\end{bmatrix}\]

Compactly this equation is again representable by,

\[(34) X^t A^t = B^t + E^t\]

\[(35) AX = B + E\]
(36) \[ AX \approx B \]

Equation (36) represents an overdetermined set of equations under the same conditions as Equation (30). The least squares solution of this equation will give estimates for the coefficients \( C_1 \) that minimize the squared error of the multichannel filter equations expressed by Equation (31).

Let the coefficient matrix \( A \) in Equation (36) be regularly partitioned in partitions \( A_{i,j} \) of dimension \( (N_b, N_o + N_i) \).

Since \( A \) stands for the transpose of the coefficient matrix in Equation (33), it is easily verified that,

(37) \[ A_{i,j} = A_{j-i}, \quad i = 1 \ldots (N_b - N_e + 1), \quad j = 1 \ldots 2p \]

The property expressed by Equation (37) indicates that \( A \) is a block Toeplitz matrix. This property will further on be used in the development of the normal equations corresponding to Equation (36).

Completely similar, overdetermined equation systems can be set up for the model with the impulse response functions, defined by Equation (2.1.75), and for the model with the free decay response data, defined by Equation (2.1.88). To obtain an overdetermined equation that can be used to estimate the coefficients \( A_1, \ldots, A_{2p} \) and \( B_0, \ldots, B_{2p-1} \) in these models, the range of \( k \) has to be selected for a pre-
windowed case. Corresponding to Equation (27), Equation (38) and (39) then follow respectively,

\[ (38) \]

\[
\begin{bmatrix}
  [y_{-1}] & [y_0] & \cdots & [y_{2p-2}] & [y_{2p-1}] & \cdots & [y_{N_e-1}] \\
  [y_{-2}] & [y_{-1}] & \cdots & [y_{2p-3}] & [y_{2p-2}] & \cdots & [y_{N_e-2}] \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  [y_{-2p}] & [y_{-2p+1}] & \cdots & [y_{-1}] & [y_0] & \cdots & [y_{N_e-2p}] \\
  I & 0 & \cdots & 0 & 0 & \cdots & 0 \\
  0 & I & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & I & 0 & \cdots & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  [y_0] & [y_1] & \cdots & [y_{2p-1}] & [y_{2p}] & \cdots & [y_{N_e}] \\
  [e_0] & [e_1] & \cdots & [e_{2p-1}] & [e_{2p}] & \cdots & [e_{N_e}]
\end{bmatrix}
\]
\[
\begin{align*}
\begin{bmatrix}
A_1 & A_2 & \ldots & A_{2p} & B_0 & B_1 & \ldots & B_{2p-1} \\
H_{-1} & H_0 & \ldots & H_{2p-2} & H_{2p-1} & \ldots & H_{N_e-1} \\
H_{-2} & H_{-1} & \ldots & H_{2p-3} & H_{2p-2} & \ldots & H_{N_e-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
H_{-2p} & H_{-2p+1} & \ldots & H_{-1} & H_0 & \ldots & H_{N_e-2p} \\
I & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & I & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I & 0 & \ldots & 0 \\
\end{bmatrix}
\end{align*}
\]

\[
= \begin{bmatrix}
H_0 & H_1 & \ldots & H_{2p-1} & H_{2p} & \ldots & H_{N_e} \\
[e_0] & [e_1] & \ldots & [e_{2p-1}] & [e_{2p}] & \ldots & [e_{N_e}] \\
\end{bmatrix}
\]

Both equations are of the form defined by Equation (28). They are overdetermined when \(N_x(N_x+1)\) is larger than \(2p(N_0+N_1)\).

Finally, Equation (38) and (39) can be written in the form of Equation (31) with \([z_j]\) defined by one of following equations,

\[
(40) \quad [z_j] = \begin{bmatrix}
[y_{j-1}] \\
R_j \\
\end{bmatrix}
\]

\[
(41) \quad [z_j] = \begin{bmatrix}
H_{j-1} \\
R_j \\
\end{bmatrix}
\]

\(R_j\) is the diagonal matrix of dimension \((N_i,N_i)\) with the unit impulse sample functions on the diagonal, as defined by \(\cdot\).
Equation (2.1.74).

2.3.2.2 General Solution for the Least Squares Problem

The least squares problem for the parameters in the time domain models, set up in previous section and summarized by Equation (30), or alternatively by Equation (36), can be solved by the same procedures that are used for the estimation of the frequency domain parameters, discussed in Section 2.2.2. These procedures can be used to find the solution \( \hat{\mathbf{x}} \) that minimizes the trace of \( \mathbf{E}^T \mathbf{E} \).

Using the normal equations method, outlined in Section 2.2.2.2, \( \hat{\mathbf{x}} \) is found as solution of,

\[
(42) \quad \mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{B}
\]

or,

\[
(43) \quad \hat{\mathbf{x}} = \mathbf{D}
\]

The coefficient matrix \( \mathbf{P} \) is symmetric and positive semidefinite. When \( \mathbf{A} \) corresponds to the transpose of the coefficient matrix in Equation (27), \( \mathbf{P} \) equals,

\[
(44) \quad \mathbf{P} = \begin{bmatrix}
\mathbf{P}_{1,1} & \mathbf{P}_{1,2} \\
\mathbf{P}_{2,1} & \mathbf{P}_{2,2}
\end{bmatrix}
\]
\[
P_1,1 = \begin{bmatrix}
R_{yy}(0) |_{N_b-1}^{N_e-1} & R_{yy}(-1) |_{N_b-1}^{N_e-1} & \cdots & R_{yy}(-2p+1) |_{N_b-1}^{N_e-1} \\
R_{yy}(0) |_{N_b-2}^{N_e-2} & R_{yy}(-1) |_{N_b-2}^{N_e-2} & \cdots & R_{yy}(-2p+2) |_{N_b-2}^{N_e-2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{yy}(0) |_{N_b-2p}^{N_e-2p} & \end{bmatrix}
\]

and,
\[
R_{yy}(i) |_{a}^{b} = \sum_{k=a}^{b} [y_k][y_{k+i}]^t
\]

\[
P_2,2 = \begin{bmatrix}
R_{xx}(0) |_{N_b}^{N_e} & R_{xx}(-1) |_{N_b}^{N_e} & \cdots & R_{xx}(-2p+1) |_{N_b}^{N_e} \\
R_{xx}(0) |_{N_b-1}^{N_e-1} & R_{xx}(-1) |_{N_b-1}^{N_e-1} & \cdots & R_{xx}(-2p+2) |_{N_b-1}^{N_e-1} \\
\vdots & \vdots & \ddots & \vdots \\
R_{xx}(0) |_{N_b-2p+1}^{N_e-2p+1} & \end{bmatrix}
\]

and,
\[
R_{xx}(i) |_{a}^{b} = \sum_{k=a}^{b} [x_k][x_{k+i}]^t
\]

\[
P_{1,2} = \begin{bmatrix}
R_{yx}(1) |_{N_b-1}^{N_e-1} & R_{yx}(0) |_{N_b-1}^{N_e-1} & \cdots & R_{yx}(-2p+2) |_{N_b-1}^{N_e-1} \\
R_{yx}(2) |_{N_b-2}^{N_e-2} & R_{yx}(1) |_{N_b-2}^{N_e-2} & \cdots & R_{yx}(-2p+2) |_{N_b-2}^{N_e-2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{yx}(2p) |_{N_b-2p}^{N_e-2p} & R_{yx}(2p-1) |_{N_b-2p}^{N_e-2p} & \cdots & R_{yx}(1) |_{N_b-2p}^{N_e-2p} \\
\end{bmatrix}
\]

and,
\[
R_{yx}(i) |_{a}^{b} = \sum_{k=a}^{b} [y_k][x_{k+i}]^t
\]

The matrix \(D\) in Equation (43) equals,

\[
(45) \quad D = \begin{bmatrix}
D_1 \\
D_2 \\
\end{bmatrix}
\]

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with,

\[
D_1 = \begin{bmatrix}
R_{yy}(1) & N_e - 1 \\
R_{yy}(2) & N_e - 2 \\
\vdots & \vdots \\
R_{yy}(2p) & N_e - 2p
\end{bmatrix}
\]

\[
D_2 = \begin{bmatrix}
R_{xy}(0) & N_e \\
R_{xy}(1) & N_e - 1 \\
\vdots & \vdots \\
R_{xy}(2p) & N_e - 2p + 1
\end{bmatrix}
\]

If \( A \) corresponds to the transpose of the coefficient matrix in Equation (33), then \( P \) equals,

\[
P = \begin{bmatrix}
R_{zz}(0) & N_e & R_{zz}(-1) & N_e & \cdots & R_{zz}(-2p+1) & N_e \\
R_{zz}(0) & N_e & R_{zz}(-1) & N_e & \cdots & R_{zz}(-2p+2) & N_e-1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R_{zz}(0) & N_e & R_{zz}(-1) & N_e & \cdots & R_{zz}(-2p+2) & N_e-1 \\
& & & & & \vdots & \vdots \\
& & & & & & R_{zz}(0) & N_e-2p+1
\end{bmatrix}
\]

(46)

with,

\[
R_{zz}(i)|^b_a = \begin{bmatrix}
R_{yy}(i)|^{b-1}_{a-1} & R_{yx}(i+1)|^{b-1}_{a-1} \\
R_{xy}(i-1)|^b_a & R_{xx}(i)|^b_a
\end{bmatrix}
\]
The corresponding matrix $D$ equals,

$$
D = \begin{bmatrix}
R_{zy}(0) |_{N_b}^{N_e} \\
R_{zy}(1) |_{N_b}^{N_e-1} \\
\vdots \\
R_{xy}(2p-1) |_{N_b}^{N_e-2p+1}
\end{bmatrix}
$$

(47)

with,

$$
R_{zy}(i) |_{a}^{b} = \begin{bmatrix}
R_{yy}(i+1) |_{a-1}^{b-1} \\
R_{xy}(i) |_{a}^{b}
\end{bmatrix}
$$

In this last equation, $P$ equals a block Toeplitz matrix that is premultiplied with its transpose. This product will in general, that is for the covariance case, not be a block Toeplitz matrix.

Alternatively the method based on the orthogonal decomposition of the coefficient matrix can be used to solve for $\hat{\mathbf{x}}$. Depending on the rank of the coefficient matrix, $\hat{\mathbf{x}}$ is found as solution of Equation (2.2.58) or as solution of Equation (2.2.62) and (2.2.69).

The coefficient matrix of the least squares problem will be full rank when the conditions specified in Section 2.3.1 are fulfilled. Hence, using the normal equations method, $P$ in Equation (43) will be full rank, and Equation (2.2.58) can be used when the method based on the orthogonal decomposition of the coefficient matrix is applied.
For the estimation of the parameters in the frequency domain models, it was argued in Section 2.2.2 and 2.2.3 that the method based on the orthogonal decomposition of the coefficient matrix should be used rather than the normal equations method. This is due to the fact that the coefficient matrix in the least squares problem will generally have a high condition number.

The coefficient matrix of the least squares problem to be solved for the parameters in the time domain models, is on the other hand normally well conditioned. To justify this, let the coefficient matrix in the least squares problem defined by Equation (30) represent the transpose of the coefficient matrix in Equation (27). Inspection of the coefficient matrix learns that all columns that contain response data will approximately have the same Euclidean length, and so do the columns that contain force input data. Particularly, when the Euclidean length of the columns that contain response data does not differ considerably from the Euclidean length of the columns that contain force input data, then the coefficient matrix A is expected to be well conditioned.

In my experience, the least squares problem for the parameters in the time domain models can be solved as well with the normal equations method as with the method based on the
orthogonal decomposition of the coefficient matrix. Computationally, the former method can easier be implemented and will also execute faster, and is therefore preferable. For certain kinds of data, this method will also yield a set of simultaneous equations that can be solved using fast recursive algorithms, as will be discussed in next section.

Finally, the time domain data that are used for estimating the parameters in the direct parameter model should be filtered to avoid aliasing errors caused by frequency components above half the sampling frequency. This is also immediately obvious when considering that in the z domain, the direct parameter model represents a rational matrix polynomial relation between the Z-transforms of the data.

2.3.2.3 Specific Solution Techniques for the Least Squares Problem

The solution of the normal equations, defined by Equation (43), with a coefficient matrix $P$ of dimension $(n,n)$ will in general require a number of multiplications proportional to $n^3$.

As indicated in the previous section, when the prediction equation is written in the form defined by Equation (31), then the coefficient matrix in the normal equations has the
special characteristic of being equal to a block Toeplitz matrix premultiplied with its transpose. The resulting product will generally not be a block Toeplitz matrix. Special algorithms have recently been developed for inverting such matrices, requiring considerable less multiplications as compared to a standard inversion [95-98]. The applicability of such algorithms has however not been examined as part of this work.

In a number of cases of practical interest, the coefficient matrix of the normal equations will become exactly equal to a block Toeplitz matrix, or may asymptotically approach a block Toeplitz matrix. When the pre- and postwindowed case can be assumed, i.e. the data are such that the range of $k$ is defined by Equation (25), then the coefficient matrix defined in Equation (46) becomes exactly equal to a block Toeplitz matrix. To verify this, the coefficient matrix of Equation (33) can be rewritten explicitly for the pre- and postwindowed case,
Using \( A \) defined in Equation (48) to calculate \( P \) in Equation (43), it then follows that \( P \) is a block Toeplitz matrix. Moreover, \( P \) is symmetric and therefore has the structure of a correlation matrix such as encountered in many standard linear prediction problems.

As mentioned before, the pre- and postwindowed case can reasonably be assumed when \( \{y_j\}_i \) represents the transient response of a causal system to a transient force input \( \{x_j\}_i \) that is applied from time zero on, and both \( \{x_j\}_i \) and \( \{y_j\}_i \) have died out in the observation period. In particular this is true when the data consist of impulse response functions or free decay response data that completely damp out in the observation period. For such data, the matrices \( [z_j] \), \( j = 0 \ldots N_t + 1 \), in Equation (48) are defined by Equation (40) and (41) respectively.

For the covariance case, the coefficient matrix \( P \) of the
normal equations can also approach asymptotically a block
Toeplitz matrix. This will be so when \( \{x_j\}_i \) and \( \{y_j\}_i \) repre-
sent stationary force input and response data of a time
invariant structure and the observation period is taken
sufficiently long, i.e. \((N_e-N_b)\) is much larger than \(2p\).
Since the structure is assumed time invariant, it follows
then that,

\[
(49) \quad R_{zz}(-j) |_{N_b}^{N_e} = R_{zz}(-j) |_{N_b-1}^{N_e-1} = \ldots = R_{zz}(-j) |_{N_b-2p+1+j}^{N_e-2p+1+j}
\]

\[ j = 0 \ldots (2p-1) \]

Therefore \( P \), defined by Equation (46), approaches a block
Toeplitz matrix.

The normal equations, defined by Equation (43), with \( P \) a
symmetric block Toeplitz matrix, can explicitly be written
as,

\[
(50)
\begin{bmatrix}
R_{zz}(0) & R_{zz}(-1) & \ldots & R_{zz}(-2p+1) \\
R_{zz}(0) & R_{zz}(-1) & \ldots & R_{zz}(-2p+2) \\
\vdots & \vdots & \ddots & \vdots \\
R_{zz}(-1) & \vdots & \ddots & R_{zz}(0)
\end{bmatrix}
\begin{bmatrix}
C_0^t \\
C_1^t \\
\vdots \\
C_{2p-1}^t
\end{bmatrix}
= 
\begin{bmatrix}
R_{zy}(0) \\
R_{zy}(1) \\
\vdots \\
R_{zy}(2p-1)
\end{bmatrix}
\]

with,

\[
R_{zz}(-j) = R_{zz}(-j) |_{N_b}^{N_e}, \quad j = 0 \ldots (2p-1)
\]

\[
R_{zy}(j) = R_{zy}(j) |_{N_b}^{N_e}, \quad j = 0 \ldots (2p-1)
\]
The blocks $R_{zz}(i)$ of the coefficient matrix are all matrices of dimension $(N_0+N_i, N_0+N_i)$. 

Equation (50) can alternatively be derived from Equation (31) as follows. Rewrite Equation (31) in transposed form,

$$
(5') \{z_k\}^t_i \cdot 0 + \ldots + \{z_{k-2p+1}\}^t_i C_{2p-1}^t + \{e_k\}^t_i = \{y_k\}^t_i
$$

$$
k = N_b \ldots N_e, \quad i = 1 \ldots N_t
$$

Premultiplying both sides with $\{z_{k-1}\}^t_i$, and after taking expectations,

$$
(52) \quad R_{zz}(1)C^t_0 + \ldots + R_{zz}(1-2p+1)C^t_{2p-1} + R_{ze}(1) = R_{zy}(1)
$$

When the prediction model is correct, then the expected cross correlation matrix $R_{ze}(1)$ can reasonably be assumed equal to 0. For a shift invariant system, all equations like Equation (52), for $l = 0 \ldots 2p-1$, form a set of collocation equations that can be solved for $C^t_0, \ldots, C^t_{2p-1}$,

$$
(53)
$$

$$
\begin{bmatrix}
R_{zz}(0) & R_{zz}(-1) & \cdots & \cdots & R_{zz}(-2p+1) \\
R_{zz}(1) & R_{zz}(0) & R_{zz}(-1) & \cdots & R_{zz}(-2p+2) \\
\vdots & & & & \vdots \\
R_{zz}(2p-2) & \cdots & \cdots & \cdots & R_{zz}(-1) \\
R_{zz}(2p-1) & R_{zz}(2p-2) & \cdots & \cdots & R_{zz}(0)
\end{bmatrix}
\begin{bmatrix}
C^t_0 \\
C^t_1 \\
\vdots \\
C^t_{2p-2} \\
C^t_{2p-1}
\end{bmatrix} = \begin{bmatrix}
\end{bmatrix}
$$

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\[
\begin{bmatrix}
R_{zy}(0) \\
R_{zy}(1) \\
\vdots \\
R_{zy}(2p-2) \\
R_{zy}(2p-1)
\end{bmatrix} =
\]

Since \( R_{zz}(j) \) equals \( R_{zz}(-j)^t \), the coefficient matrix in the latter equation is symmetric and equal to the coefficient matrix in Equation (50).

The elements of the coefficient matrix and right hand side matrix of Equation (50) can be estimated directly from the force input and response sequences, as described for Equation (46) and (47). Alternatively, with currently used digital signal processing equipment, one could choose to estimate such elements by computing the inverse Fourier transform of power spectral density estimates. Although more efficiently computed, such correlation estimates are subject to a circular effect in the calculation procedure, wrap around error, yielding biased estimates ([11], pp. 312-314, [12], pp. 68-77).

A number of algorithms have been developed that take advantage of the special structure of a symmetric block Toeplitz matrix to find recursively its inverse, or equivalently, to solve recursively a set of simultaneous linear equations with a such a matrix as coefficient matrix [99-104]. Con-
sider a symmetric block Toeplitz matrix of dimension \((n,n)\) with in general \(m\) blocks of dimension \((1,1)\), so that \(n\) equals \((ml)\). The number of multiplications required by such algorithms will be proportional to \(m^2\), as compared to \(m^3\) for conventional techniques. The number of storage locations will be of the order of \(m\), as compared to \(m^2\) for conventional methods. Substantial savings in execution time and storage can be obtained, especially when \(m\) is large.

The algorithm developed by Wiggins and Robinson [101,104] for the solution of the multichannel filter problem (a multivariate extension of the algorithm originally developed by Levinson [99]) is directly applicable and is been used as a solution method for the normal equations defined by Equation (43) when the coefficient matrix is a symmetric block Toeplitz matrix. This algorithm is also described in Appendix B.
2.3.2.4 Separate Estimation of Parameters in Homogeneous Part of the Model

For certain applications of practical interest, the desired results may be restricted to the natural frequencies, damping values and mode shapes. As indicated in Section 2.1.3, they can be calculated from the coefficients in the homogeneous part, or the autoregressive part, of the models defined by Equation (2.1.75), (2.1.87) and (2.1.88). It is therefore of interest to examine procedures that allow estimation of these coefficients separately. The coefficients in the moving average part can eventually be evaluated in a subsequent phase. Not surprisingly, such procedures are related to the Ibrahim Time Domain method, the Least Squares Complex Exponential method and the Poly-reference method, all developed using a damped complex exponential approach in Appendix A.

Consider the model defined by Equation (2.1.75) or (2.1.88), i.e. impulse response functions $H_k$ or free decay responses $[y_k]$ constitute the data. When the range of $k$ is selected for the non-windowed case defined by Equation (26), then following prediction model is valid for the matrix of impulse response functions $H_k$,
(54) \[ H_k = A_1 y_{k-1} + A_2 y_{k-2} + \ldots + A_{2p} y_{k-2p} + E_k \]
\[ \quad k = 2p \ldots N_t \]

The matrix \( E_k \) represents the prediction errors. Similarly, using free decay responses for several sets of initial conditions \([y_k]\), it follows that,

(55) \[ [y_k] = A_1 [y_{k-1}] + A_2 [y_{k-2}] + \ldots + A_{2p} [y_{k-2p}] + E_k \]
\[ \quad k = 2p \ldots N_t \]

Both equations are in the form of an autoregressive prediction model; the value of the impulse response or free decay response is predicted by its own past only.

Putting all equations like Equation (54) or (55) for \( k = 2p \ldots N_t \) together in one matrix equation, then one derives, as compared to Equation (39) and (38),

\[
\begin{bmatrix}
A_1 & A_2 & \ldots & A_{2p}
\end{bmatrix}
\begin{bmatrix}
H_{2p-1} & \ldots & H_{N_t-1} \\
H_{2p-2} & \ldots & H_{N_t-2} \\
\vdots & \ddots & \vdots \\
H_0 & \ldots & H_{N_t-2p}
\end{bmatrix}
= \begin{bmatrix}
H_{2p} & \ldots & H_{N_t}
\end{bmatrix} - \begin{bmatrix}
E_{2p} & \ldots & E_{N_t}
\end{bmatrix}
\]
\[
\begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p}
\end{bmatrix}
\begin{bmatrix}
y_{2p-1} \\
y_{2p-2} \\
\vdots \\
y_0
\end{bmatrix}
\begin{bmatrix}
y_{N_t-1} \\
y_{N_t-2} \\
\vdots \\
y_{N_t-2p}
\end{bmatrix}
= \begin{bmatrix}
y_{2p} \\
y_{N_t}
\end{bmatrix} - \begin{bmatrix}
E_2p \\
E_{N_t}
\end{bmatrix}
\]

Both Equations are of the general form,

\[(58)\quad X^tA^t = B^t + E^t\]
\[(59)\quad AX = B + E\]
\[(60)\quad AX \approx B\]

Assuming \(N_i\) reference locations or initial conditions, Equation (60) represents an overdetermined equation when \(N_i(N_t-2p+1)\) is larger than \((2pN_o)\). This equation can be solved for the least squares estimate of the model parameters using any of the methods mentioned in Section 2.3.2.2. For example, the normal equations corresponding to Equation (56) are,

\[(61)\quad \hat{P}X = D\]

with,

\[(62)\quad P =
\begin{bmatrix}
R_{HH}(0)|_{2p-1} & R_{HH}(-1)|_{2p-1} & \cdots & R_{HH}(-2p+1)|_{2p-1} \\
R_{HH}(0)|_{2p-2} & R_{HH}(-2p+2)|_{2p-2} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
R_{HH}(0)|_{N_t-2p} & \vdots & \cdots & R_{HH}(0)|_{N_t-2p}
\end{bmatrix}
\]
\[
D = \begin{bmatrix}
R_{HH}(1) & 1_{N+1}^{2p-1} \\
R_{HH}(2) & 1_{N+1}^{2p-2} \\
\vdots & \vdots \\
R_{HH}(2p) & 1_{N+1}^{0}
\end{bmatrix}
\]

and, 
\[
R_{HH}(i)_{b}^{a} = \sum_{k=a}^{b} H_{k}H_{k+i}
\]

It is worth noting that none of the specific solution techniques described in Section 2.3.2.3 apply, since for the derivation of above equations it was necessary to assume a non-windowed case for the range of k with \( N_{b} \) equal to \( 2p \).

An estimate for the coefficients in the homogeneous part of the model defined by Equation (2.1.75) or (2.1.88) can alternatively be derived as follows. Consider, for example, the model expressed by Equation (2.1.75), rewritten in transposed form as,

\[
\begin{align*}
H_{k}^t - H_{k-1}^tA_{1}^t - \ldots - H_{k-2p}^tA_{2p}^t \\
= B_{k}^tB_{0}^t + \ldots + B_{k-2p+1}^tB_{2p-1}^t
\end{align*}
\]

Premultiplying both sides with \( H_{k-1}^t \), and after taking expectations, it follows that,

\[
\begin{align*}
R_{HH}(1) - R_{HH}(1-1)A_{1}^t - \ldots - R_{HH}(1-2p)A_{2p}^t \\
= H_{-1}^tB_{0}^t + \ldots + H_{2p-1}^tB_{2p-1}^t
\end{align*}
\]

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For a causal system, it follows that, for \( l \) larger than \((2p-1)\), the above equation simplifies to,

\[
(66) \quad R_{HH}(l) - R_{HH}(l-1)A_1^t - \cdots - R_{HH}(l-2p)A_{2p}^t = 0 \\
1 > (2p-1)
\]

Equations like this latter equation, with \( l = 2p \ldots 4p-1 \), form a set of collocation equations, solvable for \( A_1, \ldots, A_{2p} \).

\[
(67) \quad \begin{bmatrix}
R_{HH}(2p-1) & R_{HH}(2p-2) & \cdots & R_{HH}(0) \\
R_{HH}(2p) & R_{HH}(2p-1) & \cdots & R_{HH}(1) \\
\vdots & \vdots & \ddots & \vdots \\
R_{HH}(4p-2) & R_{HH}(4p-3) & \cdots & R_{HH}(2p-1)
\end{bmatrix} \begin{bmatrix}
A_1^t \\
A_2^t \\
\vdots \\
A_{2p}^t
\end{bmatrix} = \begin{bmatrix}
R_{HH}(2p) \\
R_{HH}(2p+1) \\
\vdots \\
R_{HH}(4p-1)
\end{bmatrix}
\]

Note that the coefficient matrix in Equation (67) is a non-symmetric block Toeplitz matrix and that the values calculated for \( A_1, \ldots, A_{2p} \) do not represent least squares estimates.

Equation (54), conforming to Equation (A43) in Appendix A, can directly be related to the system equation used in the Ibrahim Time Domain method [30-34]. Consider hereto the following identity relations,

\[
(68) \quad H_{k-i} = H_{k-i} \quad i = 1 \ldots (2p-1)
\]

Equation (54), augmented with the \((2p-1)\) equations expressed
by Equation (68), can be written as,

(69) \[
\begin{bmatrix}
H_k \\
H_{k-1} \\
\vdots \\
H_{k-2p+1}
\end{bmatrix}
= \begin{bmatrix}
A_1 & A_2 & \cdots & A_{2p} \\
I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I
\end{bmatrix}
\begin{bmatrix}
H_{k-1} \\
H_{k-2} \\
\vdots \\
H_{k-2p}
\end{bmatrix}
+ \begin{bmatrix}
E_k \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Equation (69) can compactly be written as,

(70) \[
\tilde{H}_k = A \tilde{H}_{k-1} + \tilde{E}_k
\]

\[k = 2p \ldots N_t\]

with,

\[
\tilde{H}_i = \begin{bmatrix}
H_i \\
H_{i-1} \\
\vdots \\
H_{i-2p+1}
\end{bmatrix}, \quad \tilde{E}_i = \begin{bmatrix}
E_i \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

With \(N_i\) equal to 1, Equation (70) represents the basic equation for the Ibrahim Time Domain method. However, from the present theory it follows that this equation remains valid even when \(N_i\) differs from 1, and therefore it presents an important generalization of this method. Basically Equation (70) relates the values of impulse response functions at an instant in time to the values at the previous instant in time through a system matrix \(A\), of dimension \((2pN_o, 2pN_o)\).

Assuming the structure to be time invariant, all equations
like Equation (70) can be organized in one equation system,

\[(71) \begin{bmatrix} \tilde{H}_{2p} & \cdots & \tilde{H}_{N_t} \end{bmatrix} = A \begin{bmatrix} \tilde{H}_{2p-1} & \cdots & \tilde{H}_{N_t-1} \end{bmatrix} + [\tilde{E}_{2p} & \cdots & \tilde{E}_{N_t}]\]

Using the normal equations method, a least squares estimate for \( A \), is found from,

\[(72) \hat{P}^T = D\]

with \( P \) defined by Equation (62) and \( D \) defined by,

\[(73) D = \begin{bmatrix} \frac{R_{HH}(1)}{N_t-1} & \frac{R_{HH}(0)}{N_t-1} & \cdots & \frac{R_{HH}(\text{p}+2)}{N_t-1} \\
\frac{R_{HH}(2)}{N_t-2} & \frac{R_{HH}(1)}{N_t-2} & \cdots & \frac{R_{HH}(\text{p}+2)}{N_t-2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{R_{HH}(2p)}{N_t-2p} & \frac{R_{HH}(2p-1)}{N_t-2p} & \cdots & \frac{R_{HH}(1)}{N_t-2p} \end{bmatrix}\]

An estimate for the pole values and mode shapes can be found from the eigenvalue decomposition of \( \hat{A} \), defined by Equation (2.1.102). However, \( \hat{A} \) does not necessarily have the structure defined by Equation (2.1.101), and therefore \( \bar{V} \) will not necessarily have the structure of Equation (2.1.103). A verification of how well \( \bar{V} \) matches the structure set forth by this latter equation is proposed in the Ibrahim Time Domain method [33] as a method to sort noise modes from structure modes.

The identity relations expressed by Equation (68) do not add any new information. Therefore, the coefficients
$A_1, \ldots, A_{2p}$ can more efficiently be estimated from Equation (61) directly.

An equation like Equation (70) can similarly be developed for the free decay responses $[y_k]$ for several sets of initial conditions,

$$[\tilde{y}_k] = A[\tilde{y}_{k-1}] + \tilde{E}_k$$

$$k = 2p \ldots N_t$$

with,

$$[\tilde{y}_k] = 
\begin{bmatrix}
[y_k] \\
[y_{k-1}] \\
\vdots \\
[y_{k-2p+1}]
\end{bmatrix}$$

With $[y_k]$ representing free decay responses for one set of initial conditions, Equation (74) had first been suggested by Ibrahim. Equation (74) however includes the generalization to free decay data for several sets of initial conditions.

Having obtained an estimate for the coefficients $A_1, \ldots, A_{2p}$, for example in the model defined by Equation (2.1.75), an estimate for the coefficients $B_0, \ldots, B_{2p-1}$ of the model can be calculated recursively from the following set of equations,
\[ \hat{B}_0 = H_0 \]
\[ \hat{B}_1 = H_1 - \hat{A}_1 H_0 \]
\[ \hat{B}_2 = H_2 - \hat{A}_1 H_1 - \hat{A}_2 H_0 \]
\[ \vdots \]
\[ \hat{B}_{2p-1} = H_{2p-1} - \hat{A}_1 H_{2p-2} - \cdots - \hat{A}_{2p-2} H_0 \]

Next the modal participation factors \( L \) can be calculated from Equation (2.1.107). Alternatively, the modal participation factors can also be estimated directly, having an estimate for \( V \) and \( \hat{A} \), and using the expansion in terms of modal parameters for \( H_k \), Equation (2.1.76),
\[ H_k = \hat{V} \hat{\xi}^k L + E_k , \quad \hat{\xi} = e^{\hat{A} \Delta t} , \quad k \geq 0 \]

This expansion may include all \( 2N_m \) modes identified from the homogeneous finite difference equation. However, one has the option to include fewer modes, after eliminating obvious computational modes, such as modes with a pole value having a positive real part. Therefore, \( \hat{V}, \hat{\xi} \) and \( L \) in the latter equation should be viewed as representing the modal parameters of the modes that are included for the modal participation factor calculation.

Since the structure is assumed time invariant, Equation (76) can be written for \( k = N_b \ldots N_e \). All such equations form following equation system,
\[
\begin{bmatrix}
H_{Nb} \\
\vdots \\
H_{Ne}
\end{bmatrix} = \begin{bmatrix}
\hat{V}_{\hat{X}}^{N_{b}} \\
\vdots \\
\hat{V}_{\hat{X}}^{N_{e}}
\end{bmatrix} L + \begin{bmatrix}
E_{Nb} \\
\vdots \\
E_{Ne}
\end{bmatrix}
\]

By including sufficient data points, this equation can be made overdetermined and can be solved by the least squares method to obtain an estimate for \( L \). The normal equations corresponding to Equation (77) are,

\[
(78) \left[ \sum_{k=N_b}^{N_e} \hat{x}_k^* \hat{V}_h \hat{V}_x \hat{x}_k \right] \hat{L} = \left[ \sum_{k=N_b}^{N_e} \hat{x}_k^* \hat{V}_h H_k \right]
\]

Similar equations can be derived to estimate the coefficients \( c_1 \) or the modal participation factors directly in the model defined by Equation (2.1.88).

Consider now the models defined by Equation (2.1.82) and (2.1.89) in Section 2.1.3, i.e. using \( H_k^t \) and \([y_k]^t\) as data. Selecting \( k \) as for the non-windowed case defined by Equation (26), following equations are derived as compared to Equation (54) and (55),

\[
(79) \quad H_k^t = A_1 H_{k-1}^t + A_2 H_{k-2}^t + \ldots + A_{2p} H_{k-2p}^t + E_k \\
\quad k = 2p \ldots N_t
\]

\[
(80) \quad [y_k]^t = A_1 [y_{k-1}]^t + A_2 [y_{k-2}]^t + \ldots + A_{2p} [y_{k-2p}]^t + E_k \\
\quad k = 2p \ldots N_t
\]
In both equations, $p$ is defined by,

\[ N_m = pn_l \quad , \quad p \text{ is integer} \]

Equation (79) conforms to Equation (A21) in Appendix A. The latter equation is used in the Polyreference method [41-43]. With this method, $H_k$ is modeled as an expansion in terms of modal parameters, i.e. as a weighted superposition of characteristic solutions. Prony's algorithm is used to separate the weighting coefficients from the characteristic solutions, by generating a homogeneous, or autoregressive, finite difference equation that is solvable for the characteristic solutions. This homogeneous finite difference equation obviously should correspond to the homogeneous part of the general finite difference equation for $H_k^t$, given by Equation (2.1.82). All equations like Equation (79) or (80), for $k = 2p \ldots N_t$, form a set of simultaneous equations, overdetermined by including sufficient data and solvable by the least squares method as outlined for Equation (54) or (55). Pole values and modal participation factors are calculated from the estimates for $A_l, \ldots A_{2p}$, using Equation (2.1.110) and (2.1.111).

Obtaining an estimate for the coefficients $B_1, \ldots B_{2p-1}$ in Equation (2.1.82) or $C_1, \ldots, C_{2p-1}$ in Equation (2.1.89), after obtaining an estimate for $A_1, \ldots A_{2p}$, is done com-
pletely analogous as outlined above for the model defined by
Equation (2.1.75) or (2.1.88). The estimates can be used to
calculate an estimate for the mode shapes using Equation
(2.1.112) or (2.1.119). Eventually an estimate for the mode
shapes is obtained directly from,

(82) \( H_k^t = \hat{L}^t \hat{x} \hat{y}^t + E_k \), \( \hat{x} = e^{\Delta t} \Delta t \), \( k \geq 0 \)

As mentioned for Equation (76), \( \hat{L}, \hat{x} \) and \( \hat{V} \) may represent
modal parameters of fewer modes than are actually calculated
from the homogeneous part of the model.

Similar to Equation (75), all such equations, for
\( k = N_b \ldots N_e \), can be solved simultaneously to obtain a
least squares estimate for the mode shapes.

Consider finally the model defined by Equation (2.1.87),
rewritten in transposed form,

(83) \( \{ y_k \}_i^t - \{ y_{k-1} \}_i^t A_1^t - \ldots - \{ y_{k-2p} \}_i^t A_{2p}^t \)
\( = \{ x_k \}_i^t B_0^t + \{ x_{k-1} \}_i^t B_1^t + \ldots + \{ x_{k-2p+1} \}_i^t B_{2p-1}^t \)
\( k = N_b \ldots N_e \), \( i = 1 \ldots N_s \)

Let \( \{ x_k \}_i \) represents a random uncorrelated multivariate
white noise sequence, i.e.,

(84) \( R_{xx}(k) = 0 \), \( k \neq 0 \)

This model for the force input \( \{ x_k \}_i \) is frequently assumed
when \( \{y_k\}_i \) represents the response sequence under operating conditions where the force input can not be measured \([49-63]\). A classical technique to eliminate the unknown sequence \( \{x_k\}_i \) from the model defined by Equation (83) consists of premultiplying both sides of Equation (83) with \( \{y_k-1\}_i \) and taken expectations ([67], pp. 74-76),

\[
(85) \quad R_{yy}(1) - R_{yy}(1-1)A_1^t - \ldots - R_{yy}(1-2p)A_{2p}^t \\
= R_{xy}(1)B_0^t + R_{xy}(1-1)B_1^t + \ldots + R_{xy}(1-2p-1)B_{2p-1}^t
\]

But since \( \{y_k\}_i \) depends only on \( \{x_k\}_i \) \( \ldots \) \( \{x_k-2p+1\}_i \), it follows that, for \( l \) larger than \( (2p-1) \), Equation (85) simplifies to,

\[
(86) \quad R_{yy}(1) = R_{yy}(1-1)A_1^t + \ldots + R_{yy}(1-2p)A_{2p}^t \\
\quad l > (2p-1)
\]

Substituting \( l = 2p \ldots (4p-1) \), a set of collocation equations is obtained that can be solved exactly for the coefficients \( A_1, \ldots, A_{2p} \).

\[
(87) \\
\begin{bmatrix}
R_{YY}(2p-1) & R_{YY}(2p-2) & \ldots & R_{YY}(0) \\
R_{YY}(2p) & R_{YY}(2p-1) & \ldots & R_{YY}(1) \\
\vdots & \vdots & \ddots & \vdots \\
R_{YY}(4p-2) & R_{YY}(4p-3) & \ldots & R_{YY}(2p-1)
\end{bmatrix}
\begin{bmatrix}
A_1^t \\
A_2^t \\
\vdots \\
A_{2p}^t
\end{bmatrix}
= \\
\begin{bmatrix}
R_{YY}(2p) \\
R_{YY}(2p+1) \\
\vdots \\
R_{YY}(4p-1)
\end{bmatrix}
\]

Note that the coefficient matrix of Equation (87) is a nonsymmetric block Toeplitz matrix, and also that the matrices
$A_1, \ldots, A_{2p}$ calculated from this equation cannot be qualified as least squares estimates.

2.3.3 Use of Principal Component Response Data

In Section 2.2.3, principal component response data in the frequency domain are discussed to handle the situation where the number of response locations, $N_0$, is larger than the number of modes that are observable in the data, $N$, so that the direct parameter model becomes a large order incomplete model. The principal component response data satisfy the condition for uniqueness, discussed in Section 2.2.1, and can be used for identification of a low order complete model, that leads to an eigenvalue decomposition of order $2N$ rather than of order $2N_0$. The same concept can be used to deal with this situation for the direct parameter model identification in the time domain.

Let $\{x_k\}_i$ represent the sampled force input data at $N_1$ input locations and $\{y_k\}_i$ the corresponding response at $N_0$ output locations. Let $R_{yy}(0)$, defined by Equation (15), be of rank $N_0'$, $N_0' < N_0$. As was discussed in Section 2.3.1, then the model defined by Equation (2.1.87), with $p$ equal to 1, becomes a large order incomplete model, and its parameters
are not uniquely defined by the data. Consider the singular value decomposition of \( R_{yy}(0) \).

\[
(88) \quad R_{yy}(0) = \mathbf{U} \mathbf{S} \mathbf{U}^T
\]

\( \mathbf{S} \) is a diagonal matrix of dimension \((N_o, N_o)\) with the singular values \( s_i \) in descending order on the diagonal. \( \mathbf{U} \) is the orthogonal matrix of dimension \((N_o, N_o)\) with the corresponding singular vectors. When the rank of \( R_{yy}(0) \) equals \( N_o \), then all \( s_i \), \( i > N_o \), will be zero, or actually negligibly small. As argued in Section 2.2.3, \( N_o \) will be close to the number of modes that can be observed from the data.

Let \( \mathbf{B} \) be the matrix of dimension \((N_o', N_o)\) defined as,

\[
(89) \quad \mathbf{B} = \begin{bmatrix}
{\{\mathbf{U}\}^T_1} \\
\vdots \\
{\{\mathbf{U}\}^T_{N_o'}}
\end{bmatrix}
\]

The principal component response data, represented by \( \{\mathbf{y}'_k\}_i \) is defined as follows,

\[
(90) \quad \{\mathbf{y}'_k\}_i = \mathbf{B}\{\mathbf{y}_k\}_i
\]

Let \( \mathbf{C} \) equal the transpose of \( \mathbf{E} \), and define the series \( \{\hat{\mathbf{y}}_k\}_i \),

\[
(91) \quad \{\hat{\mathbf{y}}_k\}_i = \mathbf{C}\{\mathbf{y}'_k\}_i
\]

The series \( \{\hat{\mathbf{y}}_k\}_i \) forms an approximation of the series \( \{\mathbf{y}_k\}_i \)
such that,

\[ (92) \{e_k\}_i = \{y_k\}_i - \{\hat{y}_k\}_i = [I - CB]\{y_k\}_i \]

and,

\[ (93) \, R_{ee}(0) = \sum_{k=0}^{N_t} (\sum_{i=1}^{N_s} \{e_k\}_i^t \{e_k\}_i) = \sum_{i=N_o+1}^{N_o} s_i \{U\}_i^t \{U\}_i \]

Equation (93) indicates that the approximation of \{y_k\}_i by \{\hat{y}_k\}_i depends on the selection of \(N_o\). When \(N_o\) is selected such that the singular values \(s_j\) are all negligibly small for \(j > N_o\), then the approximation will be very good.

The principal component response data has further on following properties,

\[ (94) \, R_{y',y'}(0) = \sum_{k=0}^{N_t} (\sum_{i=1}^{N_s} \{y'_k\}_i^t \{y'_k\}_i) = S' \]

\[ (95) \, || R_{yy}(0) - R_{yy}(0) ||^2 = \sum_{j=N_o+1}^{N_o} s_j^2 \]

In Equation (94), \(S'\) is a diagonal matrix of dimension \((N_o', N_o')\) with the \(N_o'\) largest singular values in descending order on the diagonal.

Equation (88) through (95) summarize some basic results of principal component theory of vector valued variables. Represented by \(\{X_k\}_i\) and \(\{Y_k\}_i\), \(k = 1 \ldots N_f\), the discrete Fourier transform corresponding to the series \(\{x_k\}_i\) and
\{y_k\}_i, k = 0 \ldots N_t, those equations could also have been derived directly from the corresponding equations that were derived for the frequency domain data in Section 2.2.3. For example, after applying the inverse Fourier transform to Equation (2.2.74), the following equation is derived,

\begin{equation}
\{y'_k\} = \sum_{l=-\infty}^{+\infty} B_{k-l}\{y_l\}_i
\end{equation}

Herein, \(B_k\) is a diagonal matrix of dimension \((N_0, N_0)\) with the unit sample functions on the diagonal. Therefore, Equation (96) equals Equation (90). Basically this equation expresses \(\{y'_k\}_i\) as the series \(\{y_k\}_i\) filtered by the filter with impulse response function \(B_{k}\).

A parametric model such as that defined by Equation (2.1.87) can be estimated using the series \(\{x_k\}_i\) and \(\{y'_k\}_i\) as data,

\begin{equation}
A^{2p}(q^{-1})\{y'_k\}_i = B^{2p-1}(q^{-1})\{x_k\}_i
\end{equation}

\(A^{2p}(q^{-1}) = I - A_1q^{-1} - \ldots - A_{2p}q^{-2p}\)

\(B^{2p}(q^{-1}) = B_0 + B_1q^{-1} + \ldots + B_{2p-1}q^{-2p+1}\)

The order \(p\) is selected from,

\begin{equation}
N_m = pN_0, \quad N_m = N + N_c
\end{equation}

In general, a model with \(p\) in the range from 1 to 4 depending on the noise level will suffice. The parameters in the model being estimated can be used to obtain modal
parameters in a canonical response expression for the principal component response data. Estimates for the pole values and corresponding mode shapes follow from an eigenvalue decomposition of order $2N_m$, using Equation (2.1.102) and (2.1.103). An estimate for the modal participation factors is found next from Equation (2.1.107).

Representing the estimates by $\hat{\Lambda}$, $\hat{V}'$ and $\hat{L}$, a canonical response expression for $\{y'_k\}_i$ equals,

$$\{y'_k\}_i = \sum_{l=0}^{k} \hat{V}' e^{\hat{\Lambda}(k-l)\Delta t_L} \{x_l\}_i$$

(99)

From Equation (91) then,

$$\{\hat{y}_k\}_i = \sum_{l=0}^{k} \hat{V}' e^{\hat{\Lambda}(k-l)\Delta t_L} \{x_l\}_i$$

(100)

$$\{\hat{y}_k\}_i = \sum_{l=0}^{k} \hat{V} e^{\hat{\Lambda}(k-l)\Delta t_L} \{x_l\}_i$$

(101)

Since $\{\hat{y}_k\}_i$ approximates $\{y_k\}_i$, then also,

$$\{y_k\}_i = \sum_{l=0}^{k} \hat{V} e^{\hat{\Lambda}(k-l)\Delta t_L} \{x_l\}_i$$

(102)

Hence, $\hat{V}$, $\hat{\Lambda}$, and $\hat{L}$ represent estimates of the modal parameters in the canonical response expression for $\{y_k\}_i$ using $N_m$ modes, $N$ of which are structure modes and $N_o$ modes to describe noise.

When the data consist of impulse response functions or free decay responses at $N_o$ response locations for $N_i$
reference locations or initial conditions, and \( N_0 \) is larger than \( N \), then it is again advantageous to derive principal component impulse response functions or free decay responses. The pertinent equations are summarized below,

\[
(103) \quad R_{HH}(0) = \sum_{k=0}^{N_t} H_k H_k^t \\
(104) \quad R_{HH}(0) = USU^t \\
(105) \quad H_k' = BH_k \\
(106) \quad \hat{H}_k = CH_k' \\
(107) \quad R_{yy}(0) = \sum_{k=0}^{N_t} [y_k][y_k]^t \\
(108) \quad R_{yy}(0) = USU^t \\
(109) \quad [y_k'] = B[y_k] \\
(110) \quad [\hat{y}_k] = C[y_k']
\]

The matrix \( B \) in above equations is as defined by Equation (89), with \( U \) from Equation (104) or (108). The matrix \( C \) equals the matrix \( B \) transposed.

Finally, it is worth mentioning in this context that when the models defined by Equation (2.1.82) or (2.1.89) are used, then the conditions for which the parameters are uniquely defined by the data are,
(111) \( R_{H^tH}(0) = \sum_{k=0}^{N_t} H_k^t H_k \), has rank \( N_i \)

(112) \( R_{y^ty}(0) = \sum_{k=0}^{N_t} [y_k]^t[y_k] \), has rank \( N_i \)

The number of reference locations or initial conditions will generally be a lot smaller than the number of structure modes, so that the conditions expressed by above equations may be expected to be fulfilled. Therefore, instead of estimating parameters in the model defined by Equation (2.1.75) or (2.1.88) after calculation of principal component response data, alternatively the parameters in the model defined by Equation (2.1.82) or (2.1.89) can still be estimated directly. In other words, when \( N_o \) is larger than \( N \) and a model as defined by Equation (2.1.75) or (2.1.88) would become a large order incomplete model, then a model as defined by Equation (2.1.82) or (2.1.89) remains a low order complete model, assuming that \( N_i \) is smaller than \( N \).
2.4 Some Aspects of Modal Parameter Calculation, Validation and Postprocessing

The modal parameters are found as the parameters of the partial fraction expansion of the rational matrix polynomial expression, defined by Equation (2.1.51) for $H(s)$, or by Equation (2.1.98) for $H(z)$. The partial fraction expansions are based on the spectral factorization of the matrix polynomial of dimension $(N_o, N_o)$ and order $2p$ in the denominator of these expressions. In Section 2.1 it was suggested that the higher order matrix polynomial be represented as a first order matrix polynomial of dimension $(2pN_o, 2pN_o)$, and that the eigenvalues and eigenvectors be found of the companion matrix to the matrix coefficients of the higher order matrix polynomial. For the rational factor expression defined by Equation (2.1.51) the pertinent equations are Equation (2.1.56) through Equation (2.1.59), for the rational factor expression defined by Equation (2.1.98), they are Equation (2.1.100) through Equation (2.1.103).

To find the eigenvalues and eigenvectors of the companion matrix, the QR method has been used ([89] Chapter 7, [105]...
Chapter 8). This technique is known to converge on multiple eigenvalues that have a full complement of independent eigenvectors with the same speed of convergence as for single eigenvalues ([89], pp. 348, [105] pp. 520-521, pp. 540-543). This is not so for multiple eigenvalues that do not have a full complement of independent eigenvectors. However, as mentioned in Section 2.1.4, repeated modes that have a full complement of independent mode shapes is really all that can be observed from the data.

Although the companion matrix is real, it is found to be numerically more stable to treat this matrix as a complex matrix. Subroutines for calculating the eigenvalue decomposition of a general complex matrix are available in the Eispack library [106-108], distributed by the National Energy Software Center, Argonne National Laboratories, Argonne Illinois.

The parametric model that is estimated on the data, describes the structure response in terms of \( N_m \) modes. As indicated by Equation (2.2.1), \( N_m \) includes \( N_c \) complementary modes to describe the noise on the data.

It is worth noting that the pole values calculated from the spectral analysis of the rational matrix polynomial for \( H(s) \) can take on any value in the Laplace domain. In particular, even when the direct parameter model is identified using
data over a finite frequency range, from $f_{\text{min}}$ to $f_{\text{max}}$, then pole values can still be calculated for modes outside this range, helpful in describing residual effects. The frequency domain direct parameter model identification is therefore well suited to analyze data in small frequency ranges for which, as discussed in Section 2.2.4, it is also numerically better conditioned. The spectral analysis of the rational matrix polynomial for $H(z)$ will yield pole values in the $z$ domain, related to the pole values in the Laplace domain by,

$$z_i = e^{\lambda_i \Delta t}$$

From this equation follows that only pole values between 0 and $\pi/\Delta t$, or between $f_{\text{min}}$ and $(f_{\text{min}} + \pi/\Delta t)$ for a zoom range analysis, can be identified. Hence, a direct parameter model identified in the time domain cannot accommodate modes outside the range of interest that could describe residual effects. Time domain direct parameter model identification is therefore more suitable for analyzing data over a broad frequency range.

A number of guidelines and criteria can be used to select the $N$ physical structure modes from the noise modes.

In many situations, the analyst has an idea of the range of the damping ratio of the structure modes. Modes that have a damping ratio outside this range are likely candidates for
being classified as noise modes.

Having scaled all mode shapes equally, for example to unit length, then the modal participation factors will indicate modes that do contribute significantly to the response of the structure. Noise modes are anticipated to have lower values for the modal participation factors.

Assume that some of the input locations constitute a subset of the response locations. Then, as indicated in Section 2.1.4, for every combination of two input locations \( i_1 \) and \( i_2 \), \( 1 \leq i_1, i_2 \leq N_i \), that coincides with two output locations \( o_1 \) and \( o_2 \), \( 1 \leq o_1, o_2 \leq N_o \), reciprocity relations like expressed by Equation (2.1.121) for repeated modes and (2.1.122) for non-repeated modes can be verified. To make such verification practical, a normalized reciprocity ratio for repeated modes can be defined as,

\[
\min \left( \frac{|(\hat{R}_{ci})_{o1,i2}|, |(\hat{R}_{ci})_{o2,i1}|}{\max (|(\hat{R}_{ci})_{o1,i2}|, |(\hat{R}_{ci})_{o2,i1}|)} \right)_{(2)}
\]

\( \forall (i_1, i_2), \; i = 1 \ldots k \)

For non-repeated modes,

\[
\min \left( \frac{|(\hat{R}_{i})_{o1,i2}|, |(\hat{R}_{i})_{o2,i1}|}{\max (|(\hat{R}_{i})_{o1,i2}|, |(\hat{R}_{i})_{o2,i1}|)} \right)_{(3a)}
\]

or,
Above ratios are normalized between 0 and 100; a value close to 100 indicates good reciprocity between locations \( o_1 \) and \( o_2 \).

When the structure satisfies the Maxwell-Betti reciprocity principle, then for structure modes the above ratio may be expected to approach 100, except for modes that have one of the input locations close to a node. This situation is easily detectable from the modal participation factors, that will be small for such modes and input locations. For noise modes there is however no physical mechanism that will cause the above ratio to approach 100. Especially when, for every mode, several reciprocity ratios can be verified, that is, when for every mode several pairs of input locations coincide with output locations, has this criterion shown to be very helpful for identification of structure modes.

Let \( N \) structure modes be identified. When all \( N_i \) input locations are a subset of the \( N_o \) response locations, it is possible to postprocess the estimated mode shapes and modal participation factors to obtain estimates of modal mass matrices \( M_{mi} \), as defined in Section 1.3.6 by Equation
(1.3.70b). Let, for a mode of multiplicity $m_i$, $\hat{A}_i$ represent a diagonal matrix of dimension $(m_i, m_i)$ with the estimate for the pole value on the diagonal. Let $\hat{V}_{ri}$ be a matrix of dimension $(N_i, m_i)$ with, as columns, estimates of the mode shape coefficients at the $N_i$ input locations, and let $\hat{L}_i$ represent a matrix of dimension $(m_i, N_i)$ with estimates for the corresponding modal participation factors. Using such quantities, Equation (1.3.69b) can be written as,

$$\hat{L} = \hat{A}_i^{-1} \hat{V}_{ri}$$

The approximation sign is used, since the above relation would only be exact for a structure that satisfies Maxwell-Betti reciprocity. As indicated in Section 2.1.4, $m_i$ equals at most $N_i$. Therefore, Equation (4) can be solved for a least squares estimate of $\hat{A}_i$ as,

$$\hat{A}_i = \frac{\hat{V}_{ri} \hat{V}_{ri}}{\hat{L}_i \hat{V}_{ri}}$$

An estimate for the modal mass matrix $\hat{M}_{mi}$ follows using Equation (1.3.70b),

$$\hat{M}_{mi} = \frac{\hat{A}_i}{2j(\hat{A}_i)_i}$$

As a result, the mode shapes and modal participation factors are processed into a set of mode shapes and corresponding modal mass matrices, i.e. a set of modal parameters for a
structure that satisfies the reciprocity principle.

Finally, the identified model of N structure modes, eventually postprocessed to satisfy the reciprocity principle, can be validated using response prediction. The forced response is predicted using the estimated modal parameters and compared, for example graphically, with the actual measured forced response. When good correlation is found, especially for forced response with force input locations different from the ones used for the estimation of the modal parameters, then it may be concluded that the identified model of modal parameters is good.
2.5 Overview of a Modal Analysis Approach Using Direct Parameter Model Identification

The purpose of this last section is to lay out a rational modal analysis approach based on direct parameter model identification and related concepts, introduced in preceding sections.

Before doing so, it is however useful to summarize the preliminary data processing required for direct parameter model identification, as compared to what is required for modal model identification. The preprocessing for the latter models is outlined in Figure 2.3. Frequency domain methods typically require frequency response functions as data [13-17,25,26,39]. Time domain modal model identification methods require impulse response functions [30-43] and can be used to analyze free decay response data. Noteworthy though, using current digital signal processing techniques, the impulse response functions are obtained as inverse Fourier transforms of estimated frequency response functions, and are therefore subject to inherent discrete transform processing errors such as leakage, circular
effects, wrap around error, or in general truncation bias errors. Only free decay response data can be used directly with modal model identification methods; forced response data can only be used after estimation of impulse response functions.

![Diagram](image)

**Figure 2.3 Data Preprocessing, Modal Model Identification**

The data preprocessing required for direct parameter model identification is sketched in Figure 2.4. As compared to Figure 2.3, forced response data can be processed directly, reducing considerably the required preprocessing. Forced response data, after low pass filtering to avoid aliasing, can be used directly for time domain direct parameter model identification, therefore, eliminating the effect of discrete transform errors on the identification process.
Figure 2.4 Data Preprocessing, Direct Parameter Model Identification

It is also important that the data be consistent, representing the response of a time invariant structure. Testing procedures that improve the consistency, by shortening the data acquisition time and, for forced response data, allowing for simultaneous multiple inputs, are to be preferred. The development of multichannel data acquisition hardware and software, such as estimation procedures for frequency response functions from random uncorrelated simultaneous multiple inputs [19-22], can in view of this not be overstated.

Figure 2.5, 2.6 and 2.7 sketch analysis trees for direct parameter model identification in the frequency domain and time domain respectively. The data base consists of $N_5$ data
sets of forced response data at \( N_0 \) response locations for input at \( N_i \) input locations, or \( N_i \) data sets of free decay response data at \( N_0 \) response locations for \( N_i \) different sets of initial conditions. In the frequency domain every data set has data at \( N_f \) frequencies, not necessarily equally spaced, in the time domain at \((N_t+1)\) equally spaced time points. The forced response data is eventually preprocessed to frequency response functions or impulse response functions, although this is not a requirement for direct parameter model identification.

![Flowchart Diagram](image)

**Figure 2.5** Frequency Domain Direct Parameter Model Identification, Analysis Tree

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The frequency domain analysis tree, shown in Figure 2.5, is commented on first. It was discussed in Section 2.2.4, that a simultaneous analysis of data at frequencies covering a broad frequency range deteriorates the conditioning of the estimation equations. Therefore, the data will in general be analyzed in subsequent, eventually slightly overlapping, frequency ranges, especially for base band measurements. The data in the pending frequency range may be weighted so as to enhance desired information. As mentioned in Section 2.2.4, the application of such weighting affects primarily the estimates of modal damping and residues, but not very significant.

Next it is examined whether the response data over the pending analysis range represents data of a low order complete model, or a large order incomplete model. The discriminating factor is the rank $N_0^*$ of the autocorrelation matrix of lag 0 of the responses, that is determined from a singular value analysis of this matrix. In Chapter 3 some examples will be used to demonstrate how $N_0^*$ can be selected so that no relevant information is lost. When the autocorrelation matrix is not full rank, than $N_s$ sets of principal component response data are calculated.

A low order direct parameter model of order $p$ is identified next. The order $p$ is selected so that the model can accommo-
date complementary modes for describing noise and residual effects of modes outside the pending frequency range. Some aspects of selecting the order \( p \) will also be discussed in Chapter 3. Note that the frequency domain direct parameter model could be modified to include upper residuals, i.e. additional terms for describing the effect of modes at frequencies higher than the pending frequency range. The inclusion of upper residuals is found to be advantageous, reducing considerably the number of extra modes that are required for describing residual effects.

Modal parameters are calculated next from the direct parameter model. When the model was identified on principal component response data, then the calculated mode shapes are to be transformed to obtain mode shapes for the original response data. Using criteria such as outlined in Section 2.4, the structure modes are identified from the complementary modes. Particularly, the verification of reciprocity ratios as explained by Equation (2.4.2) and (2.4.3), when analyzing multiple input response data, has been found to very helpful and will be discussed in Chapter 3.

The identified modal parameters for \( N \) structure modes are then validated, primarily using response prediction, i.e. by synthesizing response data and graphically comparing it with experimental data. In general, the response data used for
the analysis can be synthesized properly. Under the assumption that the structure satisfies the reciprocity principle, it should be possible however to synthesize forced response data between any locations for which mode shape coefficients are available. When such data compare with experimental data, then the identified modal parameters form an acceptable modal model for the pending frequency range.

It may be required to iterate on $N$, the number of identified structure modes, on $p$, the order of the direct parameter model, or, when the original response data describe a large order incomplete model, on $N_0$, the number of principal component response data. For this latter case however, the number of iterations will always be restricted, since $N_0$ itself already gives an indication on the number of structure modes. Eventually, if the pending frequency range is still too broad, the analysis might have to be repeated, but using a smaller frequency range.

After identification of an acceptable set of modal parameters, the analysis is repeated for the next frequency range. The contribution of modes, identified in the previous range can be subtracted from the response data in subsequent analysis ranges [76]. This procedure then eliminates implicitly lower residuals, i.e. the influence of structure modes at frequencies lower than the frequency range under
analysis. This, along with the estimation of upper residuals in the direct parameter model, was found to improve considerably the applicability of the frequency domain direct parameter model identification.

**Figure 2.6** Time Domain Direct Parameter Model Identification, Full Model Analysis Tree

Figure 2.6 shows the analysis tree for the identification of a full, that is a non-homogeneous, direct parameter
model. The data processing flow parallels substantially the
one for the frequency domain, shown in Figure 2.5, except
for the following. With time domain direct parameter model
identification, it is not possible to describe residual
effects by modes outside the analysis range. To minimize the
influence of residuals, it is therefore important to do the
analysis on a broad frequency range. This is possible since
the numerical characteristics of the time domain method do
not deteriorate when the analysis range is broadened. The
data is digitally filtered to eliminate frequency components
outside the desired range using finite impulse response
(FIR) filters or, as when the data consist of impulse
response functions, using frequency domain processing
techniques.

The experience with weighted least squares in the frequency
domain did not justify a similar study in the time domain.
The selection of a weighting scheme that would enhance
extraction of modal parameters, would also be a lot less
convenient in the time domain than it is in the frequency
domain.

An important characteristic of the time domain approach is
that a recursive solution technique can be used for calcul-
ating the least squares estimates of the parameter in the
model, as discussed Section 2.3.2.3. The recursive solution
technique can be used when the correlation case is applica-
bable for constructing the equations, and does reduce considerably the overall solution time.

**Figure 2.7** Time Domain Direct Parameter Model Identification, Separate Estimation of Homogeneous Part and Modal Participation Factors, Analysis Tree
Finally Figure 2.7 shows the data processing flow when only the homogeneous part of the model is estimated, followed by the estimation of modal participation factors. As discussed in Section 2.3.2.4, the parameters in the homogeneous part of the model can be estimated in a least squares sense or they can be calculated from a set of collocation equations. Only the former method has been examined.

Having estimated the homogeneous part of the model, pole values and mode shapes can be calculated, and a first selection of structure modes is possible. The modal participation factors of the selected structure modes can then be estimated directly. Alternatively the parameters in the non-homogeneous part of the direct parameter model are estimated, followed by a calculation of the modal parameters as done for the identification of the full model.
Both frequency and time domain direct parameter model identification methods have been implemented and evaluated for their applicability using a super-mini computer VAX 11/780. All implementations have been interfaced with the existing "Modal-Plus" modal analysis package [109] for basic data management, data display and evaluation capabilities.

The study on direct parameter model identification in the frequency domain preceded and initiated the study on methods suitable for processing time domain data. As indicated in Chapter 2, the theoretical development of the latter methods parallels and benefits substantially from the development of the frequency domain methods. The applicability of the frequency domain methods has been examined, and some results can be found in reference 110. The frequency domain methods allow for the explicit estimation of upper residuals and for the application of frequency domain weighting schemes. Such characteristics are however outweighed by some intrinsic
numerical peculiarities, requiring special solution methods and making the frequency domain methods unattractive for a general purpose implementation. All time domain methods on the other hand, generate well conditioned equation systems. for some types of data even solvable using fast recursive solution methods. The research effort has therefore focused primarily on the applicability of time domain methods. The time domain methods have also the additional capability for processing sampled data directly, without requiring any frequency domain preprocessing.

This chapter concentrates on a discussion of various implementations of direct parameter model identification methods in the time domain, and their applicability for multiple input modal analysis. In a first section, the characteristics of the methods are discussed on analytical data. The second section deals with some results on experimental data [111].
3.1 Qualification Using Analytical Data

3.1.1 Introductory Comments

Before discussing some results obtained with analytical test data, it is useful to discuss in general the analytical data generation procedure, and some of the modal parameter validation methods used throughout this chapter.

Both impulse response functions and multiple input forced response data have been analyzed as test data. The impulse response matrix is generated using Equation (1.3.37) with, assuming a system that satisfies reciprocity, residue matrices as defined by Equation (1.3.50). Having in general $k$ modes of multiplicity $m_i$, an expression for the impulse response functions between $N_o$ response locations and $N_i$ input locations, assumed to be a subset of the response locations, equals.

\[(1a) \quad H = \sum_{i=1}^{k} V_i e^{-i^{t}A_i^{-1}v_{r_i}}\]

\[(1b) \quad H = \sum_{i=1}^{k} R_{o_i} e^{-i^{t}}\]

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(1c) \[ R_{ci} = V_i A_i^{-1} v_{ri} \]

In these expressions, \( \Lambda_i \) is a diagonal matrix of dimension \( (m_i, m_i) \) with the pole value \( \lambda_i \) on the diagonal. The corresponding independent mode shapes are organized as columns of a matrix \( V_i \) of dimension \( (N_o, m_i) \). The matrix \( V_{ri} \) is of dimension \( (N_i, m_i) \), and represents the mode shape coefficients corresponding to the \( N_i \) reference locations. \( A_i \) is a scaling matrix, symmetric and of dimension \( (m_i, m_i) \). Finally, \( R_{ci} \) represents the combined residue matrix, of dimension \( (N_o, N_i) \), as defined in Section 1.3.3. For all cases, \( H \) will be assumed to represent impulse responses for a system with real modes. For all modes, \( A_i \) is assumed equal to \( \pm j \lambda_i \). For 2N non-repeated modes, as assumed for almost all cases, above expressions simplify to,

\[
(2a) \quad H = \sum_{i=1}^{2N} \{ V \}_i e^{\lambda_i t} A_i^{-1} \{ V_r \}_i^t \\
(2b) \quad H = \sum_{i=1}^{2N} R_i e^{\lambda_i t} \\
(2c) \quad R_i = \{ V \}_i A_i^{-1} \{ V_r \}_i^t
\]

Multiple input forced response data are simulated in the time domain by convolving the impulse response functions with uncorrelated force inputs.
Frequently random noise will be added to the data. Unless specified differently, the random noise will be pseudo-random normal distributed noise. The amount of noise is specified by the ratio of the variance of the added noise to the RMS (root mean square) value of the signal; 5% RMS random noise, for example, indicates that the variance of the noise equals 5% of the RMS value of the signal. The distribution and estimated autocorrelation for the first 100 lags of a sample of 1024 time points of such a noise signal with variance of approximately .125 are shown in Figure 3.1 and 3.2.

<table>
<thead>
<tr>
<th>RANGE</th>
<th>DISTRIBUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50-0.45</td>
<td>0!+</td>
</tr>
<tr>
<td>0.45-0.40</td>
<td>0!+</td>
</tr>
<tr>
<td>0.40-0.35</td>
<td>0!+</td>
</tr>
<tr>
<td>0.35-0.30</td>
<td>0!+</td>
</tr>
<tr>
<td>0.30-0.25</td>
<td>19!++++</td>
</tr>
<tr>
<td>0.25-0.20</td>
<td>44!+++++++</td>
</tr>
<tr>
<td>0.20-0.15</td>
<td>60!++++++++++</td>
</tr>
<tr>
<td>0.15-0.10</td>
<td>96!+++++++++++++++++</td>
</tr>
<tr>
<td>0.10-0.05</td>
<td>142!+++++++++++++++++++++++</td>
</tr>
<tr>
<td>0.05-0.00</td>
<td>144!+++++++++++++++++++++++</td>
</tr>
<tr>
<td>0.00-0.05</td>
<td>150!+++++++++++++++++++++++</td>
</tr>
<tr>
<td>-0.05-0.10</td>
<td>129!+++++++++++++++++++++++</td>
</tr>
<tr>
<td>-0.10-0.15</td>
<td>126!+++++++++++++++++++++++</td>
</tr>
<tr>
<td>-0.15-0.20</td>
<td>69!+++++++++++++++++++</td>
</tr>
<tr>
<td>-0.20-0.25</td>
<td>25!++++</td>
</tr>
<tr>
<td>-0.25-0.30</td>
<td>18!+++</td>
</tr>
<tr>
<td>-0.30-0.35</td>
<td>4!+</td>
</tr>
<tr>
<td>-0.35-0.40</td>
<td>1!+</td>
</tr>
<tr>
<td>-0.40-0.45</td>
<td>0!+</td>
</tr>
<tr>
<td>-0.45-0.50</td>
<td>1!+</td>
</tr>
</tbody>
</table>

**Figure 3.1** Distribution of Pseudo-Random Normal Noise
Figure 3.2 Autocorrelation of Pseudo-Random Normal Noise (lags 0-100)

After direct parameter model identification, an expression for \( H \) is obtained in the form,

\[
(3a) \quad H = \sum_{i=1}^{k} V_i e^{\Lambda_i t L_i}
\]

\[
(3b) \quad H = \sum_{i=1}^{k} R_{ci} e^{\Lambda_i t}
\]

\[
(3c) \quad R_{ci} = V_i L_i
\]

\( L_i \) is a matrix of dimension \((m_i, N_i)\), representing the modal participation factors for mode \( i \). The multiplicity of any mode can not be larger than \( N_o \) or \( N_i \). For a system that satisfies the reciprocity principle and when all \( N_i \) input
locations are a subset of the $N_o$ response locations, then $L_i$ should be related to $V_{ri}$ by a symmetric invertible matrix $A_i$,

\[ L_i = A_i^{-1} V_{ri} \]

For the most common case of systems with only non-repeated modes, Equation (3) and (4) simplify to,

\[ H = \sum_{i=1}^{2N} \{V\}_i e^{\lambda_i t} [L]_i \]

\[ H = \sum_{i=1}^{2N} R_i e^{\lambda_i t} \]

\[ R_i = \{V\}_i [L]_i \]

\[ [L]_i = A_i^{-1} \{V\}_i \]

Reciprocity ratios, as defined in Section 2.4, will be used to verify how well the system satisfies the reciprocity principle, and will be helpful for the identification of physical modes from computational modes.

For validation of mode shapes, frequent use is made of the modal assurance criterion (MAC) value [19,12]. Having two mode shapes, $\{V\}_i$ and $\{V\}_j$ respectively, the MAC value, designated $MAC_{i,j}$, is defined as follows,

\[ MAC_{i,j} = \frac{\|V\}_i h\{V\}_j\|^2}{\sum_{i=1}^{2N} \{V\}_i h\{V\}_i (\sum_{i=1}^{2N} \{V\}_j h\{V\}_j)} \quad 0 \leq MAC_{i,j} \leq 1 \]
The MAC value is basically a correlation coefficient for linear dependence between vector \( \{V\}_i \) and \( \{V\}_j \). When MAC\(_{i,j}\) is low, near 0, this indicates no linear dependence, when MAC\(_{i,j}\) is high, near 1, this indicates a high degree of linear dependence. When the two vectors are mathematically orthogonal, then the numerator in Equation (7) equals 0, and so does the MAC value. The MAC value can be used to evaluate dependence between mode shapes obtained from one analysis, or between mode shapes obtained from different analyses.

The MAC value can also be calculated between columns of the residue matrices \( R_i \), or combined residue matrices \( R_{ci} \). With direct parameter model identification, the residue matrices are found in factored form, expressed by Equation (50), as the product of a mode shape vector and a row vector with modal participation factors. Then all columns of a residue matrix describe the same mode shape, and the MAC value between any two such columns equals exactly 1. The combined residue matrices are also found in factored form, as the product of a matrix with mode shapes and a matrix with modal participation factors. The combined residue matrix is of rank \( m_i \), and the MAC value between any two columns of the combined residue matrix may take on any value between 0 and 1.

Finally, as mentioned in the introduction, all implemen-
tations have been interfaced with the "Modal-Plus" modal analysis package [109]. In the modal data base managed by this package, modes are described by their pole value, normalized mode shape and the value and location of an element in the identified residue matrix. For non-repeated modes, and assuming reciprocity, this information is sufficient for calculating any element in the complete residue matrix. For a mode with multiplicity \( m \), such information for all corresponding mode shapes is sufficient, when the mode shapes are postprocessed to have a corresponding scaling matrix \( A \) that is diagonal.

For all analysis cases discussed in this work, the mode shapes are normalized so that the largest value of any element equals 1. The value of the largest element in the residue matrix and its location are saved, and listed in a modal parameter table, along with the natural frequency (in Hertz) and the damping ratio.

3.1.2 Applicability and Performance Comparison of Different Implementations

In the discussion on solution techniques in Section 2.3.2, it was pointed out that, in the time domain, both the method
based on the orthogonal decomposition of the coefficient matrix and the normal equations method are applicable to find the least squares estimate for the parameters in the direct parameter models. When the data is such that the pre- and postwindowed case, or correlation case, apply for building up the normal equations, then the coefficient matrix simplifies to a block Toeplitz form and the resulting equations can be solved recursively.

In a first example, some aspects of execution speed and storage requirements for the different methods are discussed and compared. In a second example, the asymptotic applicability of the recursive solution technique to process directly forced response data with random uncorrelated multiple inputs is discussed.

As test data, forced response data at 4 response locations of a 4 degrees of freedom (DOF) system are used. The characteristics of the system are summarized in Table 3.1 and 3.2. Responses and forces are assumed in the X-direction.
<table>
<thead>
<tr>
<th>MODE</th>
<th>FREQ $\omega_n$ (Hz)</th>
<th>DAMPING $\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>2</td>
<td>11.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>3</td>
<td>14.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>4</td>
<td>17.000</td>
<td>0.01000</td>
</tr>
</tbody>
</table>

Table 3.1 Pole Values, 4 DOF Analytical System, Applicability and Performance Study

<table>
<thead>
<tr>
<th>LOC</th>
<th>MODE 1</th>
<th>MODE 2</th>
<th>MODE 3</th>
<th>MODE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1X</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>2X</td>
<td>1.000</td>
<td>1.000</td>
<td>-1.000</td>
<td>-1.000</td>
</tr>
<tr>
<td>3X</td>
<td>1.000</td>
<td>-1.000</td>
<td>-1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>4X</td>
<td>1.000</td>
<td>-1.000</td>
<td>1.000</td>
<td>-1.000</td>
</tr>
</tbody>
</table>

Table 3.2 Normal, Real Mode Shapes, 4 DOF Analytical System

For the first example, 50% burst random uncorrelated forces are applied at locations 1 and 2. A typical sample of force input and response is shown in Figure 3.3. Both force and response are seen to decay out in the observation period, so that the correlation case and therefore the recursive solution technique applies exactly. For all analyses, 5 samples, each of 1024 time points, have been processed.
Figure 3.3 Force Input and Response Signal (50% Burst Random)
The normal equations, using a general solution method and a recursive solution method, and the orthogonal decomposition method, using Householder transformations and sequential accumulation in partitions of 1024 data points, have been compared. The analysis results from identification of a direct parameter model of order $p$ equal to 1, on exact analytical data, are shown in Table 3.3. The results merely confirm that for the given data all three methods are equivalent. Table 3.4 shows some analysis results from identification of a direct parameter model of order $p$ equal to 4, on data in the presence of 2.5% RMS random noise. Again the results obtained with the different methods are very similar.
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>1X+</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>1.571</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>2X+</td>
<td>3X+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>1X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Using Orthogonal Decomposition

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>2X+</td>
<td>3X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>2X+</td>
<td>1X+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>2X+</td>
<td>2X+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>1.571</td>
<td>2X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

b. Using Normal Equations, General Solution Method

c. Using Normal Equations, Recursive Solution Method

Table 3.3 Estimated Modal Parameters, Model with 4 Poles
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.01010</td>
<td>1.003</td>
<td>-1.571</td>
<td>2X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01010</td>
<td>1.004</td>
<td>1.574</td>
<td>1X+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.01012</td>
<td>1.005</td>
<td>1.571</td>
<td>2X+</td>
</tr>
<tr>
<td>16.999</td>
<td>0.01015</td>
<td>1.007</td>
<td>-1.571</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Using Orthogonal Decomposition

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.01011</td>
<td>1.003</td>
<td>-1.571</td>
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<tr>
<td>11.000</td>
<td>0.01011</td>
<td>1.004</td>
<td>1.574</td>
<td>1X+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.01011</td>
<td>1.005</td>
<td>1.571</td>
<td>2X+</td>
</tr>
<tr>
<td>16.999</td>
<td>0.01016</td>
<td>1.007</td>
<td>-1.571</td>
<td>1X+</td>
</tr>
</tbody>
</table>

b. Using Normal Equations, General Solution Method

c. Using Normal Equations, Recursive Solution Method

Table 3.4 Estimated Modal Parameters, Model with 16 Poles, 2.5% RMS Random Noise

The execution time for the three methods are compared in Tables 3.5, 3.6 and 3.7. The discriminating factor is the order p of the direct parameter model, since p controls the partitioning of the equation system to be solved for the parameters.
<table>
<thead>
<tr>
<th>ORDER p</th>
<th>NR. POLES</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>8.15*</td>
<td>10.22</td>
<td>38.52</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>15.46</td>
<td>28.05</td>
<td>105.90</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>29.60</td>
<td>88.87</td>
<td>349.70</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>59.21</td>
<td>324.70</td>
<td>1314.54</td>
</tr>
</tbody>
</table>

*: CPU time in seconds

#1, normal equations method, recursive solution

#2, normal equations method, general solution

#3, orthogonal decomposition method

Table 3.5 Execution Time Comparison, Absolute Values

<table>
<thead>
<tr>
<th>ORDER p</th>
<th>NR. POLES</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>1.90</td>
<td>2.75</td>
<td>2.75</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>3.65</td>
<td>8.70</td>
<td>9.10</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>7.25</td>
<td>31.80</td>
<td>34.15</td>
</tr>
</tbody>
</table>

#1, normal equations method, recursive solution

#2, normal equations method, general solution

#3, orthogonal decomposition method

Table 3.6 Execution Time Comparison, Normalized for Model with 4 Poles
<table>
<thead>
<tr>
<th>ORDER ( p )</th>
<th>NR. POLES</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>4</td>
<td>1</td>
<td>1.25</td>
<td>4.75</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>1</td>
<td>1.80</td>
<td>6.85</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>1</td>
<td>3.00</td>
<td>11.80</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>1</td>
<td>5.50</td>
<td>22.20</td>
</tr>
</tbody>
</table>

\( \#1 \), normal equations method, recursive solution
\( \#2 \), normal equations method, general solution
\( \#3 \), orthogonal decomposition method

**Table 3.7** Execution Time Comparison, Normalized for Recursive Solution Method

Observation of the values in Table 3.6 yields that for the recursive solution method the execution time doubles as the number of poles in the model doubles. For both other methods, the multiplication factor for the execution time is close to 3 and approaches 4 as \( p \) increases.

From Table 3.7 it follows that the orthogonal decomposition method is approximately four times slower than the general solution method for the normal equations. The ratio between the execution speed of the general solution method to the execution speed for the recursive solution method equals approximately \( 6(p+1)/10 \). This ratio is also plotted in Figure 3.4. The difference in execution speed is especially significant for larger values of \( p \).
Figure 3.4 Comparison of General Solution and Recursive Solution Method for the Normal Equations

Assuming $N_o$ response and $N_i$ force input locations, the active memory requirement for the different methods is as follows:

1. Orthogonal decomposition method, using Householder transformations and sequential accumulation in partitions of 1024 time points,

$$ - 4p^2(N_o+N_i)^2 + 2pN_o(N_o+N_i) + 1024((2p+1)N_o+2pN_i) $$

The last term in this expression is the most significant and can be reduced by sequential accumulation in smaller partitions. This will however have an adverse effect on the exe-
ution time ([88], pp. 208-212). The use of this technique in a mini-computer environment with limited memory, as currently used in most analysis systems, is therefore to date prohibitive for practical problems.

2. Normal equations, general solution method,
   
   \[-4p^2(N_0+N_i)^2 + 2pN_o(N_o+N_i)\]

3. Normal equations, recursive solution method,
   
   \[-8p(N_0+N_i)^2 + 2pN_o(N_o+N_i)\]

For most practical applications, \(N_i\) will be considerable smaller than \(N_o\); neglecting \(N_i\), the ratio of memory requirements of the general solution method to the recursive solution method approximately equals \((2p+1)/5\). This ratio is also plotted in Figure 3.4.

For the direct parameter models defined by Equation (2.1.75), (2.1.87) and (2.1.88), the order of \(p\), as defined by Equation (2.1.4), Equation (2.2.2), or for principal component response data by Equation (2.3.98), will rarely be larger than 4. The expected savings in memory requirements and execution time, when using the recursive solution method, are restricted. They are significant though for direct processing of measurement data. For the models defined by Equation (2.1.82) and (2.1.89), with \(p\) defined by Equation (2.1.83), the savings become significant.
In a second example, response data of the same system, but with random uncorrelated force input at locations 1 are analyzed using a recursive solution technique for the normal equations. As indicated in Section 2.3.2.3, the correlation case and therefore the recursive solution technique only applies asymptotically to this situation. This is also confirmed by the analysis results shown in Table 3.8 and 3.9; as the number of processed time points increases, the identified modal parameters approach the correct value. The convergence of mode shapes and natural frequencies to the correct value is also observed to be faster than for the damping values.
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.001</td>
<td>0.00671 0.9829</td>
<td>-1.538</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>11.018</td>
<td>0.01079 0.9929</td>
<td>-1.719</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>13.983</td>
<td>0.00271 1.094</td>
<td>-1.473</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>16.930</td>
<td>0.01342 1.107</td>
<td>1.539</td>
<td>1x+</td>
<td>4x+</td>
</tr>
</tbody>
</table>

a. 64 Time Points

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.002</td>
<td>0.01106 1.008</td>
<td>-1.574</td>
<td>1x+</td>
<td>3x+</td>
</tr>
<tr>
<td>11.002</td>
<td>0.01143 1.015</td>
<td>-1.582</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>13.988</td>
<td>0.01059 1.038</td>
<td>-1.553</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>17.008</td>
<td>0.01131 0.9987</td>
<td>-1.576</td>
<td>1x+</td>
<td>3x+</td>
</tr>
</tbody>
</table>

b. 256 Time Points

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.00952 0.9928</td>
<td>-1.571</td>
<td>1x+</td>
<td>3x+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.00891 0.9993</td>
<td>-1.575</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>13.999</td>
<td>0.00979 1.008</td>
<td>-1.563</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>17.001</td>
<td>0.00993 0.9993</td>
<td>-1.570</td>
<td>1x+</td>
<td>3x+</td>
</tr>
</tbody>
</table>

c. 1024 Time Points

<table>
<thead>
<tr>
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<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.00978 0.9982</td>
<td>-1.570</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.00999 1.000</td>
<td>-1.571</td>
<td>1x+</td>
<td>2x+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.00997 1.001</td>
<td>-1.571</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.01051 1.001</td>
<td>-1.570</td>
<td>1x+</td>
<td>1x+</td>
</tr>
</tbody>
</table>

d. 5120 Time Points

<table>
<thead>
<tr>
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<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.01026 0.9994</td>
<td>-1.571</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01000 1.001</td>
<td>1.570</td>
<td>1x+</td>
<td>4x+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.00999 1.001</td>
<td>-1.572</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.01006 1.000</td>
<td>-1.569</td>
<td>1x+</td>
<td>1x+</td>
</tr>
</tbody>
</table>

e. 20480 Time Points

Table 3.8 Estimated Pole Values, Random Uncorrelated Force Input, Recursive Solution Method
Table 3.9 Modal Assurance Criterion Values, Random Uncorrelated Force Input, Recursive Solution Method

The demonstrated applicability of a fast recursive solution method for direct parameter model identification, both for transient forced response data and the practical important case of response data for random uncorrelated force inputs, largely increases the potential for processing directly sampled time domain data.
3.1.3 Identification of Repeated and Pseudo-Repeated Modes

It was stressed repeatedly during the theoretical development in Chapter 2, that repeated modes with a full complement of independent mode shapes can be identified directly using direct parameter model identification. It was emphasized that this requires a simultaneous analysis of forced response data with multiple force inputs, or free decay data for several sets of initial conditions. The multiplicity of an identified repeated mode equals at most the number of independent combinations in which the corresponding mode shapes are excited from the different force input locations, or for the different sets of initial conditions.

The identification of repeated modes from test data may seem a mathematical exercise. However, in many situations, several modes may occur in clusters, to the extent that they practically appear as a repeated mode, a pseudo-repeated mode. The identification of such modes using direct parameter model identification may be expected to be possible under similar conditions as for the identification of theoretical repeated modes.

The purpose of this section is to clarify the applicability of direct parameter model identification for describing repeated and pseudo-repeated modes. Test data is generated
for a 4 DOF system with real modes, as specified in Table 3.2. For different test cases, the pole values are as listed in Table 3.10.

<table>
<thead>
<tr>
<th>REPEATED MODE SYSTEM</th>
<th>PSEUDO-REPEATED MODE SYSTEMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>FREQ ( \omega_n ) (Hz)</td>
<td>DAMPING ( \xi )</td>
</tr>
<tr>
<td>10.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>20.000</td>
<td>0.01000</td>
</tr>
</tbody>
</table>

Table 3.10 Pole Values, 4 DOF Analytical System with Repeated and Pseudo-Repeated Modes

A set of response data with a 50% burst random force input at location 1, and another set with 50% burst random uncorrelated force inputs at location 1 and 2, are generated for the system with pseudo-repeated mode of multiplicity 2. The analysis results are summarized in Table 3.11 and 3.12. As expected, since all modes are theoretically non-repeated, perfect identification is possible even for the single input analysis.
Table 3.11 Estimated Modal Parameters, Pseudo-Repeated Mode with Multiplicity 2

Table 3.12 Modal Assurance Criterion Values, Theoretical vs. Single Input Analysis, Pseudo-Repeated Mode with Multiplicity 2

Similar sets of response data are generated for the system with the repeated mode. Note that both mode shapes corresponding to the repeated modes are excited in independent combinations for force input at location 1 and 2. Analysis results are discussed in Table 3.13, 3.14 and 3.15. As expected, no identification is possible for the repeated mode from the single input response data. Table 3.13 indi-
cates that for the dual input analysis the repeated mode is however perfectly identified. The identified mode shapes, modal participation factors, combined residue matrix and scaling matrix A; calculated from Equation (4), are detailed in Table 3.14. As further shown in Table 3.15, the mode shapes are independent. The combined residue matrix is easily verified to be correct, and to satisfy the reciprocity principle between both input locations. The scaling matrix A is invertible and symmetric. Finally, Table 3.14 shows the combined residue matrix for reference 4X, calculated from the identified mode shapes and scaling matrix A.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>1X+</td>
<td>2X+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01000</td>
<td>2.000</td>
<td>1.571</td>
<td>1X+</td>
<td>3X+</td>
</tr>
<tr>
<td>17.175</td>
<td>0.73668</td>
<td>7.1359E-07</td>
<td>-0.516</td>
<td>1X+</td>
<td>2X+</td>
</tr>
<tr>
<td>20.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>1X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Single Input Analysis

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>-1.571</td>
<td>2X+</td>
<td>2X+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15.000</td>
<td>0.01000</td>
<td></td>
<td>see Table 3.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.000</td>
<td>0.01000</td>
<td>1.000</td>
<td>1.571</td>
<td>2X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

b. Dual Input Analysis

Table 3.13 Estimated Modal Parameters, Repeated Mode System

260
\[
\begin{pmatrix}
1.00000 & 0.00000 \\
0.24287 & 0.44382 \\
-1.00000 & 0.00000 \\
-0.24287 & 0.44382 \\
\end{pmatrix}
\begin{pmatrix}
1.00000 & 0.00000 \\
-0.14981 & -0.76140 \\
-1.00000 & 0.00000 \\
0.14981 & 0.76140 \\
\end{pmatrix}
\]
\[a. \text{ Mode Shapes}\]

\[
\begin{pmatrix}
0.14741 & -1.21548 \\
-0.14741 & -0.78452 \\
\end{pmatrix}
\begin{pmatrix}
-1.50019 & -0.48878 \\
1.50019 & 0.48878 \\
\end{pmatrix}
\]
\[b. \text{ Modal Participation Factors}\]

\[
\begin{pmatrix}
0.00000 & 2.00000 \\
0.00001 & -0.00001 \\
0.00000 & 2.00000 \\
-0.00001 & 0.00000 \\
\end{pmatrix}
\begin{pmatrix}
0.00000 & 0.00000 \\
-0.00001 & -1.99999 \\
0.00000 & 0.00000 \\
0.00001 & 1.99999 \\
\end{pmatrix}
\]
\[c. \text{ Combined Residue Matrix for References } 1X \text{ and } 2X\]

\[
\begin{pmatrix}
-0.55409 & 0.27801 \\
0.70151 & -1.49348 \\
\end{pmatrix}
\begin{pmatrix}
0.70150 & -1.49349 \\
-0.84892 & 0.70896 \\
\end{pmatrix}
\]
\[d. \text{ Scaling Matrix, } A^{-1}\]

\[
\begin{pmatrix}
0.00000 & 0.00000 \\
0.00001 & 1.99999 \\
0.00000 & 0.00000 \\
-0.00001 & -1.99999 \\
\end{pmatrix}
\]
\[e. \text{ Combined Residue Matrix for Reference } 4X\]

\textbf{Table 3.14} Mode Shapes, Modal Participation Factors and Intermediate Results for Repeated Mode
Table 3.15a indicates that both mode shapes, identified for the repeated mode, are independent, however not orthogonal. Table 3.15b shows the MAC values between the theoretical mode shapes, as listed in Table 3.2, and the identified mode shapes. It can be concluded that each mode identified for the repeated mode is some linear combination of both theoretical mode shapes. Since both shapes are however independent, they can be used along with the corresponding scaling matrix, alternatively to the theoretical mode shapes, for which the corresponding scaling matrix equals JJ.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
<td>0.2015</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

a. Dual Input Analysis

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>0.6933</td>
<td>0.4065</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
<td>0.3067</td>
<td>0.5935</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

b. Theoretical vs. Dual Input Analysis

Table 3.15 Modal Assurance Criterion Values, Repeated Mode System

The exercise is now repeated for the system with a pseudo-repeated mode of multiplicity 2, but with 5% RMS random
noise on the data. Analysis results are summarized in Table 3.16 and 3.17.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.998</td>
<td>0.01064</td>
<td>0.9956</td>
<td>-1.565</td>
<td>1x+</td>
<td>1x+</td>
</tr>
<tr>
<td>15.001</td>
<td>0.01064</td>
<td>2.006</td>
<td>1.562</td>
<td>1x+</td>
<td>3x+</td>
</tr>
<tr>
<td>15.104</td>
<td>0.05489</td>
<td>0.1025</td>
<td>1.745</td>
<td>1x+</td>
<td>2x+</td>
</tr>
<tr>
<td>19.998</td>
<td>0.01008</td>
<td>0.9927</td>
<td>1.577</td>
<td>1x+</td>
<td>4x+</td>
</tr>
</tbody>
</table>

a. Single Input Analysis

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.999</td>
<td>0.01001</td>
<td>0.9989</td>
<td>-1.574</td>
<td>2x+</td>
<td>1x+</td>
</tr>
<tr>
<td>14.998</td>
<td>0.01139</td>
<td>1.102</td>
<td>1.438</td>
<td>2x+</td>
<td>4x+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01012</td>
<td>1.104</td>
<td>1.428</td>
<td>1x+</td>
<td>3x+</td>
</tr>
<tr>
<td>20.001</td>
<td>0.01008</td>
<td>0.9868</td>
<td>1.574</td>
<td>1x+</td>
<td>2x+</td>
</tr>
</tbody>
</table>

b. Dual Input Analysis

Table 3.16 Estimated Modal Parameters, Pseudo-Repeated Mode with Multiplicity 2, 5% RMS Random Noise
Table 3.17  Modal Assurance Criterion Values, Pseudo-Repeated Mode with Multiplicity 2, 5% RMS Random Noise

Clearly now, the repeated mode could not be identified from the single input analysis. This is also illustrated in Figure 3.5, where the theoretical frequency response function (4X,2X) is compared with a synthesized function, using the modal parameters from the single input analysis and the modal parameters from the dual input analysis. Using the parameters of the former analysis, it is apparently impossible to have a proper synthesis around the pseudo-repeated mode.
Figure 3.5 Comparison of Theoretical and Synthesized Frequency Response Function (4X, 2X)
The same is observed when analyzing response data of the system with a pseudo-repeated mode of multiplicity 3, in the presence of 5% RMS random noise, as indicated in Table 3.18 and 3.19. Only when response data is analyzed for force input at three locations, can the repeated mode be identified.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.999</td>
<td>0.01004</td>
<td>0.9949</td>
<td>-1.569</td>
<td>1X+</td>
</tr>
<tr>
<td>15.024</td>
<td>0.02104</td>
<td>1.541</td>
<td>1.622</td>
<td>1X+</td>
</tr>
<tr>
<td>15.187</td>
<td>0.02237</td>
<td>1.795</td>
<td>1.286</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Single Input Analysis

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.999</td>
<td>0.01004</td>
<td>1.003</td>
<td>-1.571</td>
<td>2X+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.02023</td>
<td>1.060</td>
<td>-1.634</td>
<td>1X+</td>
</tr>
<tr>
<td>15.147</td>
<td>0.02136</td>
<td>2.190</td>
<td>1.535</td>
<td>1X+</td>
</tr>
<tr>
<td>15.175</td>
<td>0.03700</td>
<td>0.6591</td>
<td>0.129</td>
<td>2X+</td>
</tr>
</tbody>
</table>

b. Dual Input Analysis

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.000</td>
<td>0.01002</td>
<td>1.001</td>
<td>-1.572</td>
<td>2X+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.02029</td>
<td>1.089</td>
<td>1.545</td>
<td>1X+</td>
</tr>
<tr>
<td>15.058</td>
<td>0.02031</td>
<td>1.142</td>
<td>1.583</td>
<td>1X+</td>
</tr>
<tr>
<td>15.200</td>
<td>0.02033</td>
<td>1.084</td>
<td>-1.536</td>
<td>3X+</td>
</tr>
</tbody>
</table>

c. Triple Input Analysis

Table 3.18 Estimated Modal Parameters, Pseudo-Repeated Mode with Multiplicity 3, 5% RMS Random Noise
a. Theoretical vs. Single Input Analysis

\[
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 1.0000 & 0.0000 & 0.0000 \\
2 & 0.0000 & 0.7709 & 0.0187 \\
3 & 0.0000 & 0.2052 & 0.1735 \\
4 & 0.0000 & 0.0239 & 0.8077 \\
\end{array}
\]

b. Theoretical vs. Dual Input Analysis

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1.0000 & 0.0000 & 0.0000 & 0.0008 \\
2 & 0.0000 & 0.9991 & 0.0006 & 0.0052 \\
3 & 0.0000 & 0.0006 & 0.5212 & 0.5154 \\
4 & 0.0000 & 0.0003 & 0.4782 & 0.4786 \\
\end{array}
\]

c. Theoretical vs. Triple Input Analysis

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\
2 & 0.0000 & 0.9970 & 0.0024 & 0.0000 \\
3 & 0.0000 & 0.0029 & 0.9974 & 0.0043 \\
4 & 0.0000 & 0.0002 & 0.9998 & 0.0057 \\
\end{array}
\]

Table 3.19 Modal Assurance Criterion Values, Pseudo-Repeated Mode with Multiplicity 3, 5% RMS Random Noise

In conclusion, the above results indicate that, in the presence of noise, the identification of pseudo-repeated modes is possible with direct parameter model identification, when response data for multiple force inputs is analyzed. This characteristic will further be discussed on experimental data.
3.1.4 Influence of Random Noise

In this section, a discussion is presented of how random noise on the response data affects the identification results, and how it can be accounted for in the identification procedure. Response data of a 4 DOF system, with modes specified in Table 3.2, are used. Table 3.20 lists the corresponding pole values. A typical sample of a response signal in the presence of random noise, for a 50% burst random input at location 1, is shown in Figure 3.6.

Unless mentioned differently, all analysis results are obtained from analyzing 1 sample of 1024 time points of response data with 50% burst random force input at location 1. To accommodate for the random noise, the order of the model is taken equal to 2 or 3, depending on noise level. Table 3.21 and 3.22 indicate how identification results are influenced by different noise levels. Most sensitive are the identified damping values, with a maximum error of 3% in the presence of 10% RMS random noise.
Figure 3.6 Response Data in Presence of Random Noise, 50% Burst Random Input
<table>
<thead>
<tr>
<th>FREQ $\omega_n$ (Hz)</th>
<th>DAMPING $\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.02000</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02000</td>
</tr>
<tr>
<td>14.000</td>
<td>0.02000</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02000</td>
</tr>
</tbody>
</table>

**Table 3.20** Pole Values, 4 DOF Analytical System, Influence of Random Noise Study

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.001</td>
<td>0.02036</td>
<td>1.000</td>
<td>-1.575</td>
<td>1x+</td>
</tr>
<tr>
<td>11.005</td>
<td>0.02034</td>
<td>0.9939</td>
<td>1.555</td>
<td>1x+</td>
</tr>
<tr>
<td>13.989</td>
<td>0.02047</td>
<td>1.010</td>
<td>1.578</td>
<td>1x+</td>
</tr>
<tr>
<td>17.003</td>
<td>0.02009</td>
<td>0.9947</td>
<td>-1.572</td>
<td>1x+</td>
</tr>
</tbody>
</table>

a. 2% RMS Random Noise, Model with 8 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.003</td>
<td>0.02044</td>
<td>0.9934</td>
<td>-1.583</td>
<td>1x+</td>
</tr>
<tr>
<td>11.008</td>
<td>0.02063</td>
<td>0.9930</td>
<td>1.561</td>
<td>1x+</td>
</tr>
<tr>
<td>14.001</td>
<td>0.02056</td>
<td>1.007</td>
<td>-1.573</td>
<td>1x+</td>
</tr>
<tr>
<td>17.001</td>
<td>0.02051</td>
<td>1.010</td>
<td>-1.580</td>
<td>1x+</td>
</tr>
</tbody>
</table>

b. 5% RMS Random Noise, Model with 8 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.003</td>
<td>0.02044</td>
<td>0.9934</td>
<td>-1.583</td>
<td>1x+</td>
</tr>
<tr>
<td>11.008</td>
<td>0.02063</td>
<td>0.9930</td>
<td>1.561</td>
<td>1x+</td>
</tr>
<tr>
<td>14.001</td>
<td>0.02056</td>
<td>1.007</td>
<td>-1.573</td>
<td>1x+</td>
</tr>
<tr>
<td>17.001</td>
<td>0.02051</td>
<td>1.010</td>
<td>-1.580</td>
<td>1x+</td>
</tr>
</tbody>
</table>

c. 10% RMS Random Noise, Model with 12 Poles

**Table 3.21** Estimated Modal Parameters, Various Random Noise Levels
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9999</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.9999</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

**Table 3.22** Modal Assurance Criterion Values, Theoretical vs. Analysis. 10% RMS Random Noise

Especially in the presence of high noise levels, it may be required to increase the order of the model considerably, before a reasonable estimate of damping is achieved. This is indicated in Table 3.23, that lists some analysis results on data with 30% RMS random noise.
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.973</td>
<td>0.03270</td>
<td>1.031</td>
<td>-1.559</td>
<td>1X+</td>
</tr>
<tr>
<td>11.056</td>
<td>0.03465</td>
<td>1.062</td>
<td>-1.523</td>
<td>1X+</td>
</tr>
<tr>
<td>13.965</td>
<td>0.03478</td>
<td>1.101</td>
<td>-1.570</td>
<td>1X+</td>
</tr>
<tr>
<td>17.676</td>
<td>0.03175</td>
<td>1.049</td>
<td>1.559</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Model with 8 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.999</td>
<td>0.02429</td>
<td>1.019</td>
<td>-1.561</td>
<td>1X+</td>
</tr>
<tr>
<td>10.991</td>
<td>0.02455</td>
<td>1.026</td>
<td>1.594</td>
<td>1X+</td>
</tr>
<tr>
<td>13.993</td>
<td>0.02458</td>
<td>1.054</td>
<td>-1.547</td>
<td>1X+</td>
</tr>
<tr>
<td>17.008</td>
<td>0.02507</td>
<td>1.069</td>
<td>1.564</td>
<td>1X+</td>
</tr>
</tbody>
</table>

b. Model with 16 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.997</td>
<td>0.02205</td>
<td>1.024</td>
<td>-1.549</td>
<td>1X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.02146</td>
<td>1.019</td>
<td>1.569</td>
<td>1X+</td>
</tr>
<tr>
<td>13.995</td>
<td>0.02212</td>
<td>1.054</td>
<td>1.611</td>
<td>1X+</td>
</tr>
<tr>
<td>16.993</td>
<td>0.02251</td>
<td>1.068</td>
<td>-1.558</td>
<td>1X+</td>
</tr>
</tbody>
</table>

c. Model with 24 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.995</td>
<td>0.02143</td>
<td>1.033</td>
<td>-1.549</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02057</td>
<td>1.024</td>
<td>1.579</td>
<td>1X+</td>
</tr>
<tr>
<td>13.995</td>
<td>0.02161</td>
<td>1.057</td>
<td>-1.597</td>
<td>1X+</td>
</tr>
<tr>
<td>16.994</td>
<td>0.02101</td>
<td>1.048</td>
<td>-1.558</td>
<td>1X+</td>
</tr>
</tbody>
</table>

d. Model with 32 Poles

Table 3.23 Estimated Modal Parameters, 30% RMS Random Noise, Various Models

In all examples discussed above, a model is used that is of higher order than theoretically required, to accommodate for the presence of random noise. Along with the system modes, a number of noise modes, or computational modes, are identi-
fied; in all tables above however only the system modes have been tabulated. For the selection of system modes from computational modes, criteria such as explained in Section 2.4 are helpful. In particular, for multiple input analysis and under the assumption that the system satisfies the reciprocity principle, a verification of the reciprocity ratios, defined by Equation (2.4.2) and (2.4.3), has proved to be very useful in identifying system modes from computational modes. To demonstrate this, forced response data is analyzed for simultaneous 50% burst random uncorrelated force inputs at location 1, 2 and 3, in the presence of 10% RMS random noise. Analysis results are shown in Table 3.24.
### a. Estimated Modal Parameters

<table>
<thead>
<tr>
<th>FREQ (Hz)</th>
<th>Damping Amplitude</th>
<th>Phase</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.171</td>
<td>0.99852</td>
<td>0.509</td>
<td>1.412</td>
<td>1X+</td>
</tr>
<tr>
<td>4.950</td>
<td>0.39272</td>
<td>6.716E-02</td>
<td>1.626</td>
<td>2X+</td>
</tr>
<tr>
<td>8.004</td>
<td>0.62042</td>
<td>1.021</td>
<td>-1.591</td>
<td>2X+</td>
</tr>
<tr>
<td>8.048</td>
<td>0.26394</td>
<td>5.298E-02</td>
<td>-0.791</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02886</td>
<td>1.038</td>
<td>1.568</td>
<td>3X+</td>
</tr>
<tr>
<td>13.993</td>
<td>0.02855</td>
<td>1.025</td>
<td>-1.537</td>
<td>2X+</td>
</tr>
<tr>
<td>16.999</td>
<td>0.02662</td>
<td>1.024</td>
<td>1.581</td>
<td>3X+</td>
</tr>
<tr>
<td>17.282</td>
<td>0.11956</td>
<td>7.417E-02</td>
<td>-2.043</td>
<td>2X+</td>
</tr>
<tr>
<td>20.815</td>
<td>0.10275</td>
<td>6.377E-02</td>
<td>1.884</td>
<td>3X+</td>
</tr>
<tr>
<td>24.959</td>
<td>0.12336</td>
<td>0.1499</td>
<td>1.566</td>
<td>2X+</td>
</tr>
<tr>
<td>25.123</td>
<td>0.09883</td>
<td>5.845E-02</td>
<td>0.000</td>
<td>2X+</td>
</tr>
<tr>
<td>26.005</td>
<td>0.27526</td>
<td>0.1772</td>
<td>0.000</td>
<td>3X+</td>
</tr>
</tbody>
</table>

### b. Normalized Modal Participation Factors and Reciprocity Ratios

<table>
<thead>
<tr>
<th>FREQ (Hz)</th>
<th>Normalized Modal Participation Factors</th>
<th>Normalized Reciprocity Ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1X</td>
<td>2X</td>
</tr>
<tr>
<td>3.171</td>
<td>100</td>
<td>27</td>
</tr>
<tr>
<td>4.950*</td>
<td>43</td>
<td>100</td>
</tr>
<tr>
<td>8.004*</td>
<td>96</td>
<td>100</td>
</tr>
<tr>
<td>8.048*</td>
<td>100</td>
<td>64</td>
</tr>
<tr>
<td>11.000*</td>
<td>95</td>
<td>97</td>
</tr>
<tr>
<td>13.998*</td>
<td>98</td>
<td>100</td>
</tr>
<tr>
<td>16.999*</td>
<td>98</td>
<td>98</td>
</tr>
<tr>
<td>17.282</td>
<td>62</td>
<td>100</td>
</tr>
<tr>
<td>20.815</td>
<td>64</td>
<td>39</td>
</tr>
<tr>
<td>24.959</td>
<td>42</td>
<td>100</td>
</tr>
<tr>
<td>25.123</td>
<td>86</td>
<td>100</td>
</tr>
<tr>
<td>26.005</td>
<td>77</td>
<td>83</td>
</tr>
</tbody>
</table>

*: physical meaningful mode

Table 3.24 Identification of System Modes and Computational Modes, 10% RMS Random Noise, Model with 12 Poles

Similar results are observed when impulse response functions are used as data. A typical impulse response function, represented in the frequency domain by its corresponding frequency response function, is shown in Figure 3.7 for two types of random noise. Again, for all analysis...
results that are discussed, data at 1024 time points have been used. Table 3.25 shows some results for various levels of RMS random noise; the results are similar to what is observed in Table 3.21. Table 3.26 shows the results from identification on data with 5% peak amplitude random noise; the results are comparable with the results obtained on data with 10% RMS random noise. As an alternative, the analysis procedure where the autoregressive part is first estimated, followed by the estimation of the modal participation factors, is also been applied to the data with 5% RMS random noise. The results are summarized in Table 3.27, and are similar to the results obtained from the analysis using a full model, shown in Table 3.25b.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.002</td>
<td>0.02038</td>
<td>1.006</td>
<td>-1.574</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02064</td>
<td>1.013</td>
<td>1.568</td>
<td>2X+</td>
</tr>
<tr>
<td>14.003</td>
<td>0.02038</td>
<td>0.9978</td>
<td>-1.582</td>
<td>2X+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02034</td>
<td>1.011</td>
<td>-1.571</td>
<td>1X+</td>
</tr>
</tbody>
</table>

Table 3.26 Estimated Modal Parameters, 5% Peak Amplitude Random Noise

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.999</td>
<td>0.02052</td>
<td>1.019</td>
<td>-1.567</td>
<td>1X+</td>
</tr>
<tr>
<td>11.001</td>
<td>0.02042</td>
<td>1.014</td>
<td>-1.573</td>
<td>1X+</td>
</tr>
<tr>
<td>14.001</td>
<td>0.02046</td>
<td>1.017</td>
<td>-1.574</td>
<td>1X+</td>
</tr>
<tr>
<td>17.004</td>
<td>0.02038</td>
<td>1.013</td>
<td>1.566</td>
<td>2X+</td>
</tr>
</tbody>
</table>

Table 3.27 Estimated Modal Parameters, 5% RMS Random Noise, Estimation of Autoregressive Part of Model and Modal Participation Factors
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.02003</td>
<td>1.003</td>
<td>-1.572</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02002</td>
<td>1.002</td>
<td>-1.568</td>
<td>2X+</td>
</tr>
<tr>
<td>14.000</td>
<td>0.02015</td>
<td>1.004</td>
<td>1.570</td>
<td>2X+</td>
</tr>
<tr>
<td>17.001</td>
<td>0.02001</td>
<td>1.001</td>
<td>-1.570</td>
<td>1X+</td>
</tr>
</tbody>
</table>

**a. 2% RMS Random Noise, Model with 8 Poles**

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.02041</td>
<td>1.009</td>
<td>-1.574</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02030</td>
<td>1.006</td>
<td>-1.564</td>
<td>2X+</td>
</tr>
<tr>
<td>13.999</td>
<td>0.02064</td>
<td>1.011</td>
<td>1.568</td>
<td>2X+</td>
</tr>
<tr>
<td>17.003</td>
<td>0.02081</td>
<td>1.004</td>
<td>-1.589</td>
<td>1X+</td>
</tr>
</tbody>
</table>

**b. 5% RMS Random Noise, Model with 8 Poles**

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.02077</td>
<td>1.020</td>
<td>-1.573</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02083</td>
<td>1.013</td>
<td>-1.561</td>
<td>2X+</td>
</tr>
<tr>
<td>14.002</td>
<td>0.02090</td>
<td>1.022</td>
<td>-1.577</td>
<td>1X+</td>
</tr>
<tr>
<td>16.997</td>
<td>0.02076</td>
<td>1.020</td>
<td>1.579</td>
<td>2X+</td>
</tr>
</tbody>
</table>

**c. 10% RMS Random Noise, Model with 12 Poles**

**Table 3.25** Estimated Modal Parameters. Various Random Noise Levels
Figure 3.7 Impulse Response Functions in Presence of Random Noise
For comparison with results of other methods, it is worth observing that the effect of noise on the data, especially the relative errors on the damping ratios, are a function of the overall damping of the system. In Table 3.28 some results are compared, obtained from response data with of the same system, but with 1%, 3% and 6% damping, and with 5% RMS random noise. The maximum relative error of the damping ratio is seen to equal 5.5%, 2.5% and 1.5% respectively.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.999</td>
<td>0.01040</td>
<td>1.010</td>
<td>-1.575</td>
<td>1X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01034</td>
<td>1.007</td>
<td>-1.562</td>
<td>2X+</td>
</tr>
<tr>
<td>13.999</td>
<td>0.01055</td>
<td>1.015</td>
<td>-1.567</td>
<td>2X+</td>
</tr>
<tr>
<td>17.002</td>
<td>0.01030</td>
<td>1.004</td>
<td>-1.567</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. 1% Critical Damping

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.03039</td>
<td>1.008</td>
<td>-1.573</td>
<td>1X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.03025</td>
<td>1.005</td>
<td>-1.565</td>
<td>2X+</td>
</tr>
<tr>
<td>13.999</td>
<td>0.03073</td>
<td>1.011</td>
<td>-1.573</td>
<td>2X+</td>
</tr>
<tr>
<td>17.004</td>
<td>0.03026</td>
<td>1.004</td>
<td>-1.566</td>
<td>1X+</td>
</tr>
</tbody>
</table>

b. 3% Critical Damping

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.000</td>
<td>0.06030</td>
<td>1.006</td>
<td>-1.571</td>
<td>1X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.06008</td>
<td>1.003</td>
<td>-1.568</td>
<td>2X+</td>
</tr>
<tr>
<td>14.002</td>
<td>0.06093</td>
<td>1.013</td>
<td>-1.575</td>
<td>2X+</td>
</tr>
<tr>
<td>17.009</td>
<td>0.06021</td>
<td>1.004</td>
<td>1.564</td>
<td>4X+</td>
</tr>
</tbody>
</table>

c. 6% Critical Damping

Table 3.28 Estimated Modal Parameters, 5% RMS Random Noise, Model with 8 Poles, Various Damping Levels
In conclusion, all above examples indicate that random noise on the response data primarily affects the estimated damping values, and that the order of the model needs to be larger as the noise level increases.

3.1.5 Influence of Leakage

One of the important motives for research on direct parameter model identification is the intended use of this method for direct processing of sampled time data. This eliminates the need for frequency domain preprocessing, as outlined in Figure 2.4, and consequently inherent signal processing errors, such as leakage, circular effects, wrap around error, or in general, truncation errors caused by the discrete finite Fourier transform. The purpose of this section is to demonstrate how such errors affect the identification results when data, preprocessed in the frequency domain, is analyzed. It will be emphasized that truncation errors are typical bias errors, and unlike random noise, are not properly accounted for by increasing the order of the direct parameter model.

To simulate truncation errors, response data is simulated for a 4 DOF system with real modes, as specified in Table
3.2, and poles as specified in Table 3.29, with random force input at location 1. An ensemble of 8192 time points is generated and preprocessed in subsequent blocks of 256, 512 and 1024 time points, to estimate frequency response functions. Figure 3.8 and 3.9 show some typical frequency response and coherence functions. The truncation bias error is especially visible at the resonances of the structure and, as expected, larger when the blocksize is smaller. Figure 3.10 shows a corresponding estimate for an impulse response function, obtained as the inverse Fourier transform of an estimated frequency response function. Typically, the estimated function decays at first, but grows again towards the end of the block.

<table>
<thead>
<tr>
<th>FREQ ( \omega_n ) (Hz)</th>
<th>DAMPING ( \xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>9.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01000</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01000</td>
</tr>
</tbody>
</table>

Table 3.29 Pole Values. 4 DOF Analytical System, Influence of Leakage Study
Figure 3.8 Theoretical vs. Estimated Function, Blocksize of 512, Frequency Domain
Figure 3.9 Theoretical vs. Estimated Functions, Blocksize of 256, 512 and 1024
Figure 3.10 Theoretical vs. Estimated Function, Blocksize of 256, Time Domain
The effect of adding 5% RMS random noise on the processed time data is illustrated in Figure 3.11. Primarily a slight drop in the coherence function, especially at the antiresonances is observed.

Table 3.30 and 3.31 show the influence on the identification results when impulse response functions, obtained as inverse Fourier transform of estimated frequency response functions are analyzed with direct parameter model identification. Especially striking are the errors on damping values and residue estimates, larger for the lower frequency modes. As expected and indicated in Table 3.31, the mode shapes are not very much influenced by truncation bias errors. Note that the data for these cases was noise free, and direct processing of the sampled data using direct parameter model identification would yield exact answers.
Figure 3.11  Theoretical vs. Estimated Function, Blocksize of 512, 5% RMS Random Noise
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.997</td>
<td>0.01599 1.057</td>
<td>-1.499</td>
<td>1X+</td>
<td>3X+</td>
</tr>
<tr>
<td>8.998</td>
<td>0.01397 1.038</td>
<td>1.602</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.01319 0.9610</td>
<td>-1.575</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>12.997</td>
<td>0.01288 1.037</td>
<td>-1.582</td>
<td>1X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

a. Blocksize of 256

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.001</td>
<td>0.01204 1.020</td>
<td>-1.583</td>
<td>1X+</td>
<td>3X+</td>
</tr>
<tr>
<td>9.001</td>
<td>0.01137 1.012</td>
<td>1.547</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.01111 1.003</td>
<td>-1.575</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>12.999</td>
<td>0.01109 0.9977</td>
<td>-1.570</td>
<td>1X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

b. Blocksize of 512

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.000</td>
<td>0.01060 1.010</td>
<td>-1.544</td>
<td>1X+</td>
<td>3X+</td>
</tr>
<tr>
<td>9.000</td>
<td>0.01038 1.003</td>
<td>1.562</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.01037 0.9926</td>
<td>1.574</td>
<td>1X+</td>
<td>2X+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01020 0.9980</td>
<td>-1.575</td>
<td>1X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

c. Blocksize of 1024

Table 3.30 Estimated Modal Parameters. Various Levels of Leakage

<table>
<thead>
<tr>
<th>1</th>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3.31 Modal Assurance Criterion Values, Theoretical vs. Analysis, Blocksize of 256
Unlike random noise, the truncation error cannot be accounted for by increasing the order of the model. This is illustrated in Table 3.32. Very typical, when the order of the model is increased, the modes split up and appear as a pseudo-repeated mode, as indicated in Table 3.32c. In particular, the estimated residue values become completely erroneous. From the MAC values, calculated for the modes corresponding to the poles in Table 3.32c, it follows that the mode shapes corresponding to such computational repeated modes are really identical, as shown in Table 3.33.
### a. Model with 4 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.001</td>
<td>0.01204</td>
<td>1.020</td>
<td>-1.583</td>
<td>1X+</td>
<td>3X+</td>
</tr>
<tr>
<td>9.001</td>
<td>0.01137</td>
<td>1.012</td>
<td>1.547</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.01111</td>
<td>1.003</td>
<td>-1.575</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>12.999</td>
<td>0.01109</td>
<td>0.9977</td>
<td>-1.570</td>
<td>1X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

### b. Model with 8 Poles

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.998</td>
<td>0.01285</td>
<td>1.059</td>
<td>-1.559</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>8.990</td>
<td>0.01257</td>
<td>1.063</td>
<td>1.677</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>10.999</td>
<td>0.01129</td>
<td>1.010</td>
<td>1.574</td>
<td>1X+</td>
<td>2X+</td>
</tr>
<tr>
<td>12.999</td>
<td>0.01154</td>
<td>1.025</td>
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<td>1X+</td>
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</table>

### c. Model with 12 Poles

<table>
<thead>
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<th>DAMPING</th>
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<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
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<td>6.948</td>
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<tr>
<td>7.958</td>
<td>0.01064</td>
<td>0.4866</td>
<td>-1.799</td>
<td>1X+</td>
<td>1X+</td>
</tr>
<tr>
<td>8.947</td>
<td>0.08682</td>
<td>0.4449</td>
<td>1.754</td>
<td>1X+</td>
<td>1X+</td>
</tr>
<tr>
<td>9.945</td>
<td>0.08351</td>
<td>0.5734</td>
<td>1.672</td>
<td>1X+</td>
<td>4X+</td>
</tr>
<tr>
<td>10.919</td>
<td>0.01137</td>
<td>0.3118</td>
<td>-1.151</td>
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<td>1X+</td>
</tr>
<tr>
<td>11.033</td>
<td>0.01148</td>
<td>0.7228</td>
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<td>4X+</td>
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<td>12.997</td>
<td>0.01030</td>
<td>0.2857</td>
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<td>1X+</td>
</tr>
<tr>
<td>13.037</td>
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<td>0.7309</td>
<td>-1.696</td>
<td>1X+</td>
<td>3X+</td>
</tr>
</tbody>
</table>

### Table 3.32 Estimated Modal Parameters, Blocksize of 512, Various Models

### Table 3.33 Modal Assurance Criterion Values, Blocksize of 512, Model with 12 Poles

288
Table 3.34 lists the results obtained when applying direct parameter model identification on impulse response functions, after estimation of frequency response functions on the data with 5% RMS random noise and processed in blocks of 256. Table 3.35 shows the results from processing the same data directly, using a recursive solution method. For both cases, the order of the model equals 2, i.e. a model with 8 poles is estimated on the data. The results from the latter approach are clearly superior.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.998</td>
<td>0.01584</td>
<td>1.065</td>
<td>-1.508</td>
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</tr>
<tr>
<td>9.000</td>
<td>0.01386</td>
<td>1.836</td>
<td>1.598</td>
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</tr>
<tr>
<td>10.997</td>
<td>0.01322</td>
<td>0.9747</td>
<td>-1.573</td>
<td>1x+</td>
</tr>
<tr>
<td>12.997</td>
<td>0.01291</td>
<td>1.046</td>
<td>-1.553</td>
<td>1x+</td>
</tr>
</tbody>
</table>

Table 3.34 Estimated Modal Parameters, Frequency Domain Preprocessing, Blocksize of 256, 5% RMS Random Noise

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.001</td>
<td>0.01013</td>
<td>1.005</td>
<td>-1.581</td>
<td>1x+</td>
</tr>
<tr>
<td>8.999</td>
<td>0.01067</td>
<td>1.004</td>
<td>-1.574</td>
<td>1x+</td>
</tr>
<tr>
<td>11.002</td>
<td>0.01050</td>
<td>1.007</td>
<td>-1.569</td>
<td>1x+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01033</td>
<td>1.006</td>
<td>-1.571</td>
<td>1x+</td>
</tr>
</tbody>
</table>

Table 3.35 Estimated Modal Parameters, Direct Processing, 5% RMS Random Noise

In conclusion, when impulse response functions, obtained from inverse Fourier transforming estimated frequency response functions, are used, then inherent frequency domain
processing errors are reflected in the analysis results. Such errors have the character of a bias error and are not necessarily reduced by increasing the order of the model. The elimination of such errors is however possible by direct time domain processing; in view of such applications, the applicability of a fast recursive solution method for the equation systems generated with direct parameter model identification can not be overstated.

3.1.6 Use of Principal Component Response Data

As outlined in Section 2.2. and 2.3, the direct parameter model cannot uniquely be identified from the data when $N_o$, the number of response locations, is much larger than $N$, the number of modes that are observable from the data; that is when the direct parameter model becomes a large order incomplete model. It was suggested that principal component analysis could be used to approximate the original response data at the $N_o$ response locations by principal component response data at $N'_o$ equivalent locations ($N'_o$ approximately equal to $N$) that then can be used to estimate a low order complete model. The quality of the approximation depends on the value of $N'_o$, determined as the breakpoint between the significant and insignificant singular values of the auto-
correlation matrix for lag 0 of the original response data. Among the properties of the principal component response data, especially the property expressed by Equation (2.3.94) is useful. This equation expresses that the autocorrelation matrix for lag 0 of the principal component response data is diagonal, with the singular values of the autocorrelation matrix for lag 0 of the original data on the diagonal. Principal component response data that correspond to large singular values represent, therefore, more energy and more information than response data corresponding to small singular values.

The purpose of this section is to clarify the use of principal component response data for direct parameter model identification, specifically on systems that have local and pseudo-repeated modes. As data, impulse response functions, generated in the time domain, at 40 response locations of a 7 DOF system are used. The pole values for three different cases are indicated in Table 3.36. Five modes are global modes, defined by Equation (8), the two remainder modes are local modes, as defined by Table 3.37. Figure 3.12 shows the frequency response function, corresponding to a typical impulse response function, for the general system in the presence of 5% RMS random noise; the local modes are indicated.
<table>
<thead>
<tr>
<th>GENERAL SYSTEM</th>
<th>REPEATED MODE</th>
<th>PSEUDO-REPEATED MODE</th>
</tr>
</thead>
<tbody>
<tr>
<td>FREQ ( \omega_n ) (Hz)</td>
<td>DAMPING ( \xi )</td>
<td>FREQ ( \omega_n ) (Hz)</td>
</tr>
<tr>
<td>7.000</td>
<td>0.02000</td>
<td>9.000</td>
</tr>
<tr>
<td>10.000</td>
<td>0.02000</td>
<td>9.000</td>
</tr>
<tr>
<td>11.000*</td>
<td>0.02000</td>
<td>11.000*</td>
</tr>
<tr>
<td>13.000*</td>
<td>0.02000</td>
<td>13.000*</td>
</tr>
<tr>
<td>15.000*</td>
<td>0.02000</td>
<td>15.000*</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02000</td>
<td>18.000</td>
</tr>
<tr>
<td>19.000</td>
<td>0.02000</td>
<td>18.000</td>
</tr>
</tbody>
</table>

*: local mode

Table 3.36 Pole Values, 7 DOF Analytical System with Local Modes

\[(8) \sin(2\pi ki/41), \quad k = 1 \ldots 5 \text{ (modes),} \]
\[i = 1 \ldots 40 \text{ (locations)}\]

<table>
<thead>
<tr>
<th>LOC</th>
<th>11.000 Hz</th>
<th>LOC</th>
<th>15.000 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>5X</td>
<td>1.0</td>
<td>8X</td>
<td>0.2</td>
</tr>
<tr>
<td>8X</td>
<td>0.2</td>
<td>13X</td>
<td>-1.0</td>
</tr>
<tr>
<td>17X</td>
<td>-1.0</td>
<td>14X</td>
<td>1.0</td>
</tr>
<tr>
<td>20X</td>
<td>0.2</td>
<td>15X</td>
<td>-1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20X</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 3.37 Definition of Local Modes
Figure 3.12 Typical Frequency Response Function (8X, 20X), System with Local Modes, 5% RMS Random Noise

A set of impulse response functions for reference location 8X and 20X have been generated for the general system. The singular values of the autocorrelation matrix for lag 0 of the original data are listed in Table 3.38, using data for one reference location and using data for two reference locations. A breakpoint is clearly visible after 7 singular values, as expected. Accumulated singular values, normalized in % values and eventually plotted as shown in Figure 3.13, are helpful for the identification of the breakpoint. It may also be observed that of all 7 significant singular values, the first 5 are a lot larger than the last 2, characteristic for the presence of local modes.
<table>
<thead>
<tr>
<th>NR.</th>
<th>REF 8X VALUE</th>
<th>%</th>
<th>REFS 8X, 20X VALUE</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.79E+3</td>
<td>42</td>
<td>0.86E+3</td>
<td>35</td>
</tr>
<tr>
<td>2</td>
<td>0.48E+3</td>
<td>68</td>
<td>0.61E+3</td>
<td>59</td>
</tr>
<tr>
<td>3</td>
<td>0.47E+3</td>
<td>93</td>
<td>0.50E+3</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>69.</td>
<td>97</td>
<td>0.26E+3</td>
<td>91</td>
</tr>
<tr>
<td>5</td>
<td>38.</td>
<td>99</td>
<td>0.21E+3</td>
<td>99</td>
</tr>
<tr>
<td>6</td>
<td>3.1</td>
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<td>99</td>
</tr>
<tr>
<td>7</td>
<td>2.5</td>
<td>100</td>
<td>5.0</td>
<td>99</td>
</tr>
<tr>
<td>8</td>
<td>0.20E-2</td>
<td>100</td>
<td>0.22E-2</td>
<td>100</td>
</tr>
<tr>
<td>9</td>
<td>0.99E-3</td>
<td>.</td>
<td>0.19E-2</td>
<td>.</td>
</tr>
<tr>
<td>10</td>
<td>0.80E-3</td>
<td>.</td>
<td>0.16E-2</td>
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</tr>
<tr>
<td>11</td>
<td>0.72E-3</td>
<td>.</td>
<td>0.47E-3</td>
<td>.</td>
</tr>
<tr>
<td>12</td>
<td>0.20E-3</td>
<td>.</td>
<td>0.39E-3</td>
<td>.</td>
</tr>
<tr>
<td>13</td>
<td>0.10E-3</td>
<td>.</td>
<td>0.31E-3</td>
<td>.</td>
</tr>
</tbody>
</table>

*: accumulated singular values, normalized in % values

Table 3.38 Singular Values, System with Local Modes

**Principal Component Analysis/Response Data**

<table>
<thead>
<tr>
<th>NR</th>
<th>%</th>
<th>ACCUMULATED SINGULAR VALUES (0-100 SCALE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>******************************************</td>
</tr>
<tr>
<td>2</td>
<td>59</td>
<td>******************************************</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>******************************************</td>
</tr>
<tr>
<td>4</td>
<td>91</td>
<td>******************************************</td>
</tr>
<tr>
<td>5</td>
<td>99</td>
<td>******************************************</td>
</tr>
<tr>
<td>6</td>
<td>99</td>
<td>******************************************</td>
</tr>
<tr>
<td>7</td>
<td>99</td>
<td>******************************************</td>
</tr>
<tr>
<td>8</td>
<td>100</td>
<td>******************************************</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>******************************************</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>******************************************</td>
</tr>
<tr>
<td>11</td>
<td>100</td>
<td>******************************************</td>
</tr>
<tr>
<td>12</td>
<td>100</td>
<td>******************************************</td>
</tr>
<tr>
<td>13</td>
<td>100</td>
<td>******************************************</td>
</tr>
</tbody>
</table>

Figure 3.13 Singular Values, Normalized in % Values, System with Local Modes
In the presence of random noise, the breakpoint becomes less visible, as is indicated in Table 3.39 for various noise levels. Especially for higher noise levels and with the presence of local modes, one could decide the breakpoint to be at 5 rather than 7. To resolve this ambiguity, explicit inspection of the principal component response data is advisable. For the 5% RMS random noise case, the principal component response data for reference 8X corresponding to singular values 4, 7 and 8 in Table 3.39 are plotted in Figure 3.14. Noteworthy from Figure 3.14a and 3.14b is that the principal component response data is not less noisy than the original data, requiring therefore additional modes in the direct parameter model. The modal density is also seen to be reduced, similar to the effect obtained with enhanced frequency response function methods [19]. Figure 3.14b indicates that information is still present in the principal component response corresponding to singular value 7, as compared to the principal component response corresponding to singular value 8, shown in Figure 3.14c, which has a merely random characteristic. The proper breakpoint therefore is at 7.
<table>
<thead>
<tr>
<th>NR.</th>
<th>2% RMS NOISE VALUE</th>
<th>%</th>
<th>5% RMS NOISE VALUE</th>
<th>%</th>
<th>10% RMS NOISE VALUE</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.86E+3</td>
<td>35</td>
<td>0.86E+3</td>
<td>35</td>
<td>0.86E+3</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>0.61E+3</td>
<td>59</td>
<td>0.61E+3</td>
<td>59</td>
<td>0.61E+3</td>
<td>59</td>
</tr>
<tr>
<td>3</td>
<td>0.50E+3</td>
<td>80</td>
<td>0.50E+3</td>
<td>80</td>
<td>0.50E+3</td>
<td>79</td>
</tr>
<tr>
<td>4</td>
<td>0.26E+3</td>
<td>91</td>
<td>0.26E+3</td>
<td>90</td>
<td>0.26E+3</td>
<td>90</td>
</tr>
<tr>
<td>5</td>
<td>0.21E+3</td>
<td>99</td>
<td>0.21E+3</td>
<td>99</td>
<td>0.21E+3</td>
<td>98</td>
</tr>
<tr>
<td>6</td>
<td>6.3</td>
<td>99</td>
<td>6.4</td>
<td>99</td>
<td>7.0</td>
<td>98</td>
</tr>
<tr>
<td>7</td>
<td>5.0</td>
<td>99</td>
<td>5.1</td>
<td>99</td>
<td>5.5</td>
<td>99</td>
</tr>
<tr>
<td>8</td>
<td>0.41E-1</td>
<td>99</td>
<td>0.26</td>
<td>99</td>
<td>1.0</td>
<td>99</td>
</tr>
<tr>
<td>9</td>
<td>0.37E-1</td>
<td>99</td>
<td>0.23</td>
<td>99</td>
<td>0.93</td>
<td>99</td>
</tr>
<tr>
<td>10</td>
<td>0.37E-1</td>
<td>99</td>
<td>0.23</td>
<td>99</td>
<td>0.93</td>
<td>99</td>
</tr>
<tr>
<td>11</td>
<td>0.35E-1</td>
<td>99</td>
<td>0.22</td>
<td>99</td>
<td>0.87</td>
<td>99</td>
</tr>
<tr>
<td>12</td>
<td>0.34E-1</td>
<td>99</td>
<td>0.21</td>
<td>99</td>
<td>0.85</td>
<td>99</td>
</tr>
<tr>
<td>13</td>
<td>0.33E-1</td>
<td>99</td>
<td>0.21</td>
<td>99</td>
<td>0.84</td>
<td>99</td>
</tr>
</tbody>
</table>

*: accumulated singular values, normalized in % values

Table 3.39 Singular Values, Dual Input Analysis, Various Random Noise Levels

![Principal Component Response Data](image)

a. Principal Component Response, 4
Figure 3.14 Principal Component Responses for Reference 8X, System with Local Modes, Dual Input Analysis, 5% RMS Random Noise
Table 3.40 lists some results for the system with 2 repeated modes, and for the system with 2 pseudo-repeated modes. For the repeated mode system, apparently only 5 modes are observable from one reference location, while 7 modes can be observed from both reference locations. With two pseudo-repeated modes and 2% RMS random noise, the list of singular values is very much similar for both the single and dual reference analysis. The suggested breakpoint is 7, as also confirmed by some principal component response data plotted in Figure 3.15.

<table>
<thead>
<tr>
<th>REPEATED MODE SYSTEM</th>
<th>PSEUDO-REPEATED MODE SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2% RMS NOISE</td>
</tr>
<tr>
<td>REF 8x</td>
<td>REFS 8x, 20x</td>
</tr>
<tr>
<td>NN.</td>
<td>VALUE</td>
</tr>
<tr>
<td>1</td>
<td>0.12E+4</td>
</tr>
<tr>
<td>2</td>
<td>0.48E+3</td>
</tr>
<tr>
<td>3</td>
<td>0.10E+3</td>
</tr>
<tr>
<td>4</td>
<td>3.2</td>
</tr>
<tr>
<td>5</td>
<td>2.6</td>
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<tr>
<td>6</td>
<td>0.58E-2</td>
</tr>
<tr>
<td>7</td>
<td>0.30E-2</td>
</tr>
<tr>
<td>8</td>
<td>0.15E-2</td>
</tr>
<tr>
<td>9</td>
<td>0.64E-3</td>
</tr>
<tr>
<td>10</td>
<td>0.22E-3</td>
</tr>
<tr>
<td>11</td>
<td>0.14E-3</td>
</tr>
</tbody>
</table>

*: accumulated singular values, normalized in % values

Table 3.40 Singular Values, System with Local, Repeated, and Pseudo-Repeated Modes
**Figure 3.15** Principal Component Responses for Reference 8x, System with Local and Pseudo-Repeated Modes, Single Input Analysis, 2% RMS Random Noise
After identification of principal component response data, a low order direct parameter model is fitted to obtain modal parameters. Table 3.41 and 3.42 show the results obtained from fitting a model with 7 poles to the principal component response data corresponding to the 7 largest singular values in Table 3.38, using both references. The modal parameters, including the mode shapes of the original data are exactly identified from the low order model. Note that the off-diagonal terms in Table 3.42 are due to the non-orthogonality (in the mathematical sense) of the original mode shapes, and are not caused as a result of erroneous identification.

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMP</th>
<th>AMPL</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.000</td>
<td>0.01999</td>
<td>0.6365</td>
<td>-1.571</td>
<td>8x+</td>
<td>5x+</td>
</tr>
<tr>
<td>10.000</td>
<td>0.02000</td>
<td>0.9823</td>
<td>-1.571</td>
<td>8x+</td>
<td>12x+</td>
</tr>
<tr>
<td>11.000</td>
<td>0.02000</td>
<td>0.2600</td>
<td>1.571</td>
<td>8x+</td>
<td>17x+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01999</td>
<td>0.8730</td>
<td>1.571</td>
<td>8x+</td>
<td>12x+</td>
</tr>
<tr>
<td>15.000</td>
<td>0.01999</td>
<td>0.2000</td>
<td>-1.571</td>
<td>8x+</td>
<td>14x+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02000</td>
<td>0.5757</td>
<td>1.571</td>
<td>20x+</td>
<td>32x+</td>
</tr>
<tr>
<td>19.000</td>
<td>0.01999</td>
<td>0.6938</td>
<td>1.571</td>
<td>20x+</td>
<td>1x+</td>
</tr>
</tbody>
</table>

**Table 3.41** Estimated Modal Parameters, Dual Input Analysis, 7 Principal Component Responses, Model with 7 Poles

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0914</td>
<td>0.0000</td>
<td>0.0135</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0102</td>
<td>0.0000</td>
<td>0.0085</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.0914</td>
<td>0.0102</td>
<td>1.0000</td>
<td>0.0227</td>
<td>0.0010</td>
<td>0.0360</td>
</tr>
<tr>
<td>4</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0227</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>0.0135</td>
<td>0.0085</td>
<td>0.0010</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0041</td>
</tr>
<tr>
<td>6</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0360</td>
<td>0.0000</td>
<td>0.0041</td>
<td>1.0000</td>
</tr>
<tr>
<td>7</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0012</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

**Table 3.42** Modal Assurance Criterion Values, Theoretical vs. Dual Input Analysis, 7 Principal Component Responses, Model with 7 Poles
With noise on the data, the errors tend to be more pronounced for the local modes, especially visible in the damping values and residue estimates as listed in Table 3.43. Note that to accommodate for the noise on the data, the order of the model has been increased. The identification of the mode shapes remains almost perfect, also for the local modes, as indicated in Table 3.44. To improve the estimates for the damping values of the local modes, the direct parameter model identification procedure is reapplied but using only response data at the locations 5, 13, 14, 15 and 17, where the local modes are located. A slight improvement is possible, as indicated in Table 3.45.
<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.999</td>
<td>0.02008 0.6375</td>
<td>1.570</td>
<td>8X+</td>
<td>36X+</td>
</tr>
<tr>
<td>9.999</td>
<td>0.02031 0.9829</td>
<td>1.572</td>
<td>8X+</td>
<td>23X+</td>
</tr>
<tr>
<td>11.004</td>
<td>0.02100 0.2027</td>
<td>1.563</td>
<td>8X+</td>
<td>17X+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01930 0.8794</td>
<td>1.571</td>
<td>8X+</td>
<td>12X+</td>
</tr>
<tr>
<td>15.001</td>
<td>0.02063 0.2017</td>
<td>1.569</td>
<td>8X+</td>
<td>13X+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02029 0.5773</td>
<td>-1.570</td>
<td>8X+</td>
<td>9X+</td>
</tr>
<tr>
<td>18.997</td>
<td>0.02002 0.6951</td>
<td>1.573</td>
<td>8X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

a. Analysis Using 7 Principal Component Responses, Model with 14 Poles. 2% RMS Random Noise

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.000</td>
<td>0.02012 0.6395</td>
<td>1.571</td>
<td>8X+</td>
<td>36X+</td>
</tr>
<tr>
<td>10.001</td>
<td>0.02007 0.9829</td>
<td>1.568</td>
<td>8X+</td>
<td>23X+</td>
</tr>
<tr>
<td>11.017</td>
<td>0.02612 0.2038</td>
<td>1.553</td>
<td>8X+</td>
<td>17X+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.01935 0.8795</td>
<td>1.570</td>
<td>8X+</td>
<td>12X+</td>
</tr>
<tr>
<td>14.997</td>
<td>0.02438 0.2081</td>
<td>1.575</td>
<td>8X+</td>
<td>13X+</td>
</tr>
<tr>
<td>17.000</td>
<td>0.02013 0.5774</td>
<td>-1.569</td>
<td>8X+</td>
<td>9X+</td>
</tr>
<tr>
<td>18.999</td>
<td>0.02006 0.6946</td>
<td>1.573</td>
<td>8X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

b. Analysis Using 10 Principal Component Responses, Model with 20 Poles. 5% RMS Random Noise

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.999</td>
<td>0.02030 0.6427</td>
<td>1.573</td>
<td>8X+</td>
<td>36X+</td>
</tr>
<tr>
<td>10.001</td>
<td>0.01993 0.9820</td>
<td>1.569</td>
<td>8X+</td>
<td>23X+</td>
</tr>
<tr>
<td>11.008</td>
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<td>8X+</td>
<td>17X+</td>
</tr>
<tr>
<td>13.000</td>
<td>0.02062 0.8819</td>
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<td>8X+</td>
<td>12X+</td>
</tr>
<tr>
<td>15.648</td>
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<td>8X+</td>
<td>13X+</td>
</tr>
<tr>
<td>16.998</td>
<td>0.02019 0.5776</td>
<td>-1.563</td>
<td>8X+</td>
<td>9X+</td>
</tr>
<tr>
<td>19.000</td>
<td>0.02020 0.6957</td>
<td>1.572</td>
<td>8X+</td>
<td>1X+</td>
</tr>
</tbody>
</table>

c. Analysis Using 10 Principal Component Responses, Model with 30 Poles. 10% RMS Random Noise

Table 3.43 Estimated Modal Parameters, Using Principal Component Response Data, Various Random Noise Levels
Table 3.44 Modal Assurance Criterion Values, Theoretical vs. Dual Input Analysis, 10 Principal Component Responses, Model with 20 Poles, 5% RMS Random Noise

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0530</td>
<td>0.0000</td>
<td>0.0130</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
<td>1.0000</td>
<td>0.0826</td>
<td>0.0000</td>
<td>0.0091</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0371</td>
<td>0.0104</td>
<td>0.9959</td>
<td>0.0226</td>
<td>0.0013</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
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<td>0.0000</td>
<td>0.2228</td>
<td>1.0000</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0032</td>
<td>0.0009</td>
<td>0.0000</td>
<td>0.9953</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
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<td>0.0000</td>
<td>0.0352</td>
<td>0.0000</td>
<td>0.0049</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>7</td>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0014</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 3.45 Estimated Modal Parameters, Separate Analysis of Local Modes, Model with 20 Poles, 5% RMS Random Noise

<table>
<thead>
<tr>
<th>FREQ</th>
<th>DAMPING</th>
<th>AMPLITUDE</th>
<th>PHASE</th>
<th>REF</th>
<th>RES</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.999</td>
<td>0.0020</td>
<td>0.1536</td>
<td>1.581</td>
<td>20X+</td>
<td>5X+</td>
</tr>
<tr>
<td>10.003</td>
<td>0.0004</td>
<td>0.3000</td>
<td>1.564</td>
<td>20X+</td>
<td>13X+</td>
</tr>
<tr>
<td>11.022</td>
<td>0.0245</td>
<td>0.2070</td>
<td>-1.689</td>
<td>20X+</td>
<td>5X+</td>
</tr>
<tr>
<td>12.998</td>
<td>0.0205</td>
<td>0.4424</td>
<td>-1.578</td>
<td>20X+</td>
<td>5X+</td>
</tr>
<tr>
<td>15.003</td>
<td>0.0297</td>
<td>0.2039</td>
<td>1.561</td>
<td>20X+</td>
<td>13X+</td>
</tr>
<tr>
<td>17.000</td>
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The above results indicate that, even in the presence of local and pseudo-repeated modes, the principal component response data can be used for identification of a low order direct parameter model to obtain estimates for modal parameters. A key step is the selection of \( N_0' \), the number of principal component responses to be used. Especially in the presence of local and pseudo-repeated modes, this requires a critical examination of the principal component response data, in addition to the evaluation of the singular values of the autocorrelation matrix for lag 0 of the original response data.
3.2 Experimental Cases

In this section, two experimental test cases are discussed which demonstrate the applicability of direct parameter model identification for multiple input modal analysis of mechanical structures.

For both cases, the data consist of measured frequency response functions for several reference locations. For every response location, the frequency response functions are measured simultaneously for all reference locations, so that good consistency exists between the data for all reference locations. Both cases deal with a mechanical structure that exhibits highly coupled pseudo-repeated modes, and therefore represent good examples to demonstrate the capability for extracting such modes using direct parameter model identification.

The frequency response functions are measured in acceleration over force format. Although the direct parameter model identification methods in the time domain were formulated for data in displacement over force format, it is possible to use the same formulae for analyzing data in acceleration.
over force format, since the data in either format has the characteristic of a superposition of damped complex exponentials. Therefore, the data will not be transformed to displacement over force format, unless indicated explicitly.

All analyses have also been done using a non-recursive solution method. The applicability of the recursive solution method will however be discussed. The remarks in Section 3.1.1 on validation procedures for the identified modal parameters remain also valid in this section.

3.2.1 Analysis of a Symmetric Structure

A symmetric structure will have a number of pseudo-repeated modes and therefore represents a good test case to demonstrate the capability for extracting closely spaced, nearly repeated modes. As test article a metal circular plate with outer diameter of approximately 18 inches has been used. The same structure has been used before to demonstrate the capabilities to extract closely spaced, pseudo-repeated modes with the Polyreference method [41-43] and has also been used as test article for the frequency domain direct parameter model identification methods [83,110].
Frequency response functions have been measured for 7 reference locations and 36 response locations, all indicated in Figure 3.16. The frequency response functions, of dimension acceleration over force, have been measured using impact testing and 7 reference accelerometers simultaneously (located at the 7 reference locations) to obtain consistent measurements between the different reference locations. The particular analysis system consisted of a Hewlett-Packard 5451B Fourier Analyzer using a 8 channel Zonic 6080 Multi-channel FFT Analyzer as data acquisition unit. The frequency range of interest is from 200 Hz to 1480 Hz, with a frequency resolution of 5 Hz. Figure 3.17 shows a typical driving point frequency response function, indicating 5 modes of
which however are found to be repeated with multiplicity of 2.

Figure 3.17 Typical Driving Point Frequency Response Function, (1Y-, 1Y)

Several analyses, using different sets of reference locations, various analysis methods, and both the data in acceleration over force format as well as data transformed to displacement over force format have been done.

A first analysis is performed using as data frequency response functions for reference location 1Y-. Note that the data that is actually used in the analysis are the corre-
sponding impulse response functions. The singular values, normalized in $\%$ values, are listed in the first column of Table 3.46. The break point in the singular values appears at 5, indicating that approximately 5 modes are observable from the data. A direct parameter model of order $p$ equal to 2 is fitted to 12 principal component responses. Of all 24 modes, 9 were identified as physical meaningful and their modal parameters are tabulated in Table 3.47. Apparently 4 modes of multiplicity 2 are identified. However the mode shapes corresponding to some of these modes are very much correlated, as observable from the M$\&$C values listed in Table 3.48, and illustrated in Figure 3.18 for the two modes around 760 Hz. The mode shape coefficients of the mode at 761.599 Hz are nearly the opposite of the mode shape coefficients of the mode at 763.996 Hz. The defective identification of the mode shapes corresponding to the modes of multiplicity 2 is further confirmed from comparing synthesized frequency response functions with experimental frequency response functions, as in Figure 3.19.
### REFERENCES

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*: accumulated singular values normalized in % values

#1, ref. 1Y-
#2, refs. 1Y-, 6Y-
#3, refs. 6Y-, 12Y-
#4, refs. 1Y-, 8Y-, 17Y-
#5, refs. 1Y-, 6Y-, 8Y-, 17Y-, 22Y-

Table 3.46 Singular Values, Various Analyses for Circular Plate

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Table 3.47 Estimated Modal Parameters, Analysis Using Reference 1Y-
Figure 3.18 Mode Shapes for Repeated Mode at 760 Hz, Analysis Using Reference 1Y-
Table 3.48 Modal Assurance Criterion Values,
Analysis Using Reference 1Y-
c. Driving Point Functions at 4 Radial Symmetric Locations

Figure 3.19 Comparison of Experimental and Synthesized Frequency Response Functions, Analysis Using Reference 1Y-
The analysis is now repeated, but using simultaneously frequency response functions for reference locations 1Y- and 6Y-. The normalized singular values of the autocorrelation matrix of the responses are listed as the second column in Table 3.46. The breakpoint is now observed at 9, as compared to 5 for the previous analysis, indicating that at least 4 more modes are observable from the data. Table 3.49 shows the modal parameters of 9 physical meaningful modes, identified from fitting a direct parameter model of order p equal to 2 to 16 principal component responses. Again 4 modes of multiplicity 2 are identified. The corresponding mode shapes are also independent, as observable from the table with MAC values, Table 3.50, and illustrated in Figure 3.20 for the modes at approximately 760 Hz. A net improvement is also observed when synthesizing frequency response functions using the modal parameters identified from this analysis, as in Figure 3.21.

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Table 3.49 Estimated Modal Parameters, Analysis Using References 1Y-, 6Y-
a. 761.286 Hz Mode Shape

b. 764.218 Hz Mode Shape

Figure 3.20 Mode Shapes for Repeated Mode at 760 Hz, Analysis Using References 1Y-, 6Y-
Table 3.50 Modal Assurance Criterion Values, Analysis Using References 1Y-, 6Y-

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CIRCLES: EXPERIMENTAL CURVE
LINE: SYNTHESIZED CURVE
ANALYSIS USING REFERENCES 1Y-, 6Y-

a. (1Y-, 11Y)
c. Driving Point Functions at 4 Radial Symmetric Locations

Figure 3.21 Comparison of Experimental and Synthesized Frequency Response Functions, Analysis Using References 1Y-, 6Y-
The above results indicate that the mode shapes corresponding to the modes of multiplicity 2 are excited in independent combinations from both reference locations 1Y- and 6Y-. Only then is it possible to identify 2 independent mode shapes for a mode of multiplicity 2. To clarify this, the analysis is repeated, but using frequency response functions for two diametrically opposed reference locations, 6Y- and 12Y-. The break point in the normalized singular values of the autocorrelation matrix, listed in the third column of Table 3.46, is again observed at 5, as for the analysis using data for reference location 1Y- only. Modal parameters for 9 modes, identified from a direct parameter model of order p equal to 2 and fitted to 12 principal component responses, are tabulated in Table 3.51. Table 3.52 indicates that the mode shapes corresponding to the modes of multiplicity 2 are much more correlated as compared to the previous analysis, and Figure 3.22 shows a poor synthesis. Although both analyses, using data for 2 reference locations, use an equal amount of data and therefore have equal cost, the results of the latter analysis are clearly inferior to the ones obtainable from the former analysis, caused by a less optimum choice of the reference locations.
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Table 3.51 Estimated Modal Parameters, Analysis Using References 6Y-, 12Y-

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Table 3.52 Modal Assurance Criterion Values, Analysis Using References 6Y-, 12Y-
Figure 3.22 Comparison of Experimental and Synthesized Frequency Response Functions, (8Y-, 10Y-), Analysis Using References 6Y-, 12Y-

As a next test case, frequency response functions for 3 reference locations, 1Y-, 8Y- and 17Y-, are analyzed simultaneously. The singular values of the autocorrelation matrix of the responses are listed as the fourth column in Table 3.46, and indicate that the same number of modes is observable as for the analysis using data for references 1Y- and 6Y-. Some principal component responses are also shown in Figure 3.23.
Figure 3.23 Principal Component Response Data for Reference 8Y-, Analysis Using References 1Y-, 8Y- and 17Y-
Modal parameters for 9 modes, identified from fitting a direct parameter model of order 7 equal to 2 to 16 principal component responses, are shown in Table 3.53. This table, the MAC values in Table 3.54 and some synthesized frequency response functions plotted in Figure 3.24, all indicate that the identified modal parameters are equivalent to the ones obtained from the analysis using data for reference locations 1Y- and 6Y-. In a sense, this analysis then confirms that the 4 repeated modes do have a multiplicity of 2.

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Table 3.53 Estimated Modal Parameters, Analysis Using References 1Y-, 8Y- and 17Y-

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Table 3.54 Modal Assurance Criterion Values, Analysis Using References 1Y- and 6Y- vs. Analysis Using References 1Y-, 8Y- and 17Y-
a. \((1Y-, 11Y)\)

b. \((6Y-, 4Y)\)
c. Driving Point Functions at 4 Radial Symmetric Locations

Figure 3.24 Comparison of Experimental and Synthesized Frequency Response Functions, Analysis Using References $1Y_-$, $8Y_-$ and $17Y_-$

The 9 modes, whose modal parameters are listed in Table 3.53, were picked out as physical meaningful modes among all modes identified with the direct parameter model. The modal parameters of all modes, along with the normalized modal participation factors and normalized reciprocity ratios are tabulated in Table 3.55. Most modes in this table can be disregarded as computational modes, based on their high damping ratio. An exception is the mode at 1220.023 Hz, which has a damping ratio very comparable to the damping ratio of the neighboring modes at 1223.020 Hz and 1224.069 Hz. This mode can however be disregarded based on the low
value of reciprocity ratios. This table also demonstrates that for a proper interpretation of the reciprocity ratios, one should consider the normalized modal participation factors. For example, both reference locations 1Y- and 17Y- are clearly at a node of the mode at 764.186 Hz and therefore the calculated reciprocity ratios, be they low or high, are not very significant.
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<th>FREQUENCY (Hz)</th>
<th>DAMPING</th>
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<th>PHASE</th>
<th>REF</th>
<th>RES</th>
<th>NORMALIZED MODAL PARTICIPATION FACTORS</th>
<th>NORMALIZED RECIPROCITY RATIOS</th>
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*: physical meaningful mode
The frequency response functions for the three references 1Y-, 8Y- and 17Y- have also been analyzed using some alternative analyses. Table 3.56 and 3.57 show modal parameters and MAC values of an analysis using a recursive solution method, as compared to the general solution method, and of an analysis using a separate estimation of the homogeneous part of the model followed by an estimation of modal participation factors. All results are comparable to the results obtained with previous analysis.

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a. Recursive Solution Method

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b. Estimation of Homogeneous Part of Model, Followed by Estimation of Modal Participation Factors

Table 3.56 Estimated Modal Parameters, Analysis Using References 1Y-, 8Y- and 17Y-
a. Recursive Solution Method

b. Estimation of Homogeneous Part of Model, Followed by Estimation of Modal Participation Factors

Table 3.57 Modal Assurance Criterion Values, Analysis Using References 1Y- and 6Y- vs. Analysis Using References 1Y-, 8Y- and 17Y-

All analyses so far have been done on data in acceleration over force format. The direct parameter model identification models in the time domain were derived for data in displacement over force format but can, as mentioned before, be used for analysis of data in acceleration over force format, since the data in either format consists fundamentally of a superposition of damped complex exponentials. To demonstrate this equivalence, an analysis is done on frequency response functions for references 1Y-, 8Y- and 17Y-, after transfor-
mation to displacement over force format. Analysis results, from fitting a direct parameter model of order p equal to 2 to 16 principal component responses, are shown in Table 3.58, 3.59 and Figure 3.25, and are completely comparable with the results shown in Table 3.53, 3.54 and Figure 3.24. Note that the residue values listed in Table 3.58 are also in displacement over force format and equal approximately the residue values listed in Table 3.53, in acceleration over force format, after multiplication with the corresponding natural frequency squared.

<table>
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Table 3.58 Estimated Modal Parameters, Analysis Using References 1Y-, 8Y- and 17Y-, Displacement over Force Format

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</table>

Table 3.59 Modal Assurance Criterion Values, Analysis Using References 1Y- and 6Y- vs. Analysis Using References 1Y-, 8Y- and 17Y-, Displacement over Force Format
a. \((1Y-, 11Y)\)

b. \((6Y-, 4Y)\)
c. Driving Point Functions at 4 Radial Symmetric Locations

Figure 3.25 Comparison of Experimental and Synthesized Frequency Response Functions, Analysis Using References 1Y-, 8Y- and 17Y-, Displacement over Force Format

The use of data in acceleration over force format tends to weight slightly the higher frequency modes, as compared to using data in displacement over force format that tends to weight the lower frequency modes. This is demonstrated in Table 3.60, showing modal parameters obtained from fitting a direct parameter model of order \( p \) equal to 1 to 16 principal component responses, using both types of data. With the data in acceleration over force format, the estimated damping values of the lower frequency modes are erroneous while with the data in displacement over force format the damping estimates of the higher frequency modes are off. After
increasing the order of the model to 2, good estimates are possible for all modes using either data format, as demonstrated by previous analyses.

<table>
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<tr>
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<td>1Y-</td>
<td>5Y+</td>
</tr>
<tr>
<td>764.161</td>
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<tr>
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a. Acceleration over Force Format

<table>
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</table>

b. Displacement over Force Format

Table 3.60 Estimated Modal Parameters. Analysis Using References 1Y-, 8Y- and 17Y-, Model with 16 Poles

As a final test case, frequency response functions have been analyzed for 5 reference locations simultaneously, 1Y-, 6Y-, 8Y-, 17Y- and 22Y-. The normalized singular values are listed as the last column in Table 3.46. Again the breakpoint is observed at 9. Some analysis results are listed in Table 3.61, 3.62 and illustrated in Figure 3.26. All re-
results are comparable to the analysis results using data for 2 or 3 reference locations, and confirm again that in the given frequency range 4 pseudo-repeated modes with multiplicity 2 are present.

<table>
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Table 3.61 Estimated Modal Parameters, Analysis Using References 1Y-, 6Y-, 8Y-, 17Y- and 22Y-

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Table 3.62 Modal Assurance Criterion Values, Analysis Using References 1Y- and 6Y- vs. Analysis Using References 1Y-, 6Y-, 8Y-, 17Y- and 22Y-
Figure 3.26 Synthesized Driving Point Frequency Response Functions at 4 Radial Symmetric Locations, Analysis Using References 1Y-, 6Y-, 8Y-, 17Y- and 22Y-
3.2.2 Analysis of Aircraft Structure

As a second test case, direct parameter model identification is used for the modal analysis of an aircraft structure. Typically, an aircraft structure will have a number of closely coupled symmetric and anti-symmetric modes, that may be hard to detect with standard experimental modal analysis techniques. The aircraft is a General Dynamics F-16, tested as a contract requirement on Air Force Contract F08635-80-C-0166 [113] by the Structural Dynamics Research Laboratory, Department of Mechanical and Industrial Engineering of the University of Cincinnati. The aircraft was tested using random uncorrelated dual inputs for various symmetric configurations of the two input locations, and all response data were processed to frequency response functions [19-22]. Modal parameters have been estimated using classical modal model identification methods, including the Least Squares Complex Exponential method [39,40,113]. Closely spaced symmetric and anti-symmetric modes could be resolved using such methods when frequency response functions, obtained from adding and subtracting frequency response functions corresponding to symmetric input locations, are analyzed.

Using direct parameter model identification, it will be possible to identify such closely spaced modes by analyzing simultaneously frequency response functions for both refer-
ence locations. The data set that is discussed contains frequency response functions at 78 locations, unidirectional, for two input locations, symmetrically located at the wingtips. The frequency response functions were processed for the zoom range from 4.25 Hz to 16.75 Hz with a frequency resolution of 0.0244 Hz. A schematic of the airplane and both input locations are illustrated in Figure 3.27. Figure 3.28 shows a typical driving point frequency response function. A coupled symmetric and anti-symmetric mode were found to be present at 8.1 Hz.

Figure 3.27 Schematic of General Dynamics F-16 Aircraft

#: input location
Figure 3.28 Typical Driving Point Frequency Response Function (61Z,61Z)

All frequency response functions have been processed simultaneously, for both reference locations and for the global frequency range at once. The results of the principal component analysis are shown in Figure 3.29. Figure 3.30 shows some typical principal component response data. A total of 14 principal component responses for each input location have been used to fit a direct parameter model of order \( p \) equal to 2, that is a model for 28 poles.
Figure 3.29 Singular Values, Normalized in % Values, Analysis Using References 62 and 61Z

a. Principal Component Response, 3
Figure 3.30 Principal Component Response Data for Reference 6Z, Analysis Using References 6Z and 61Z
Table 3.63 shows the modal parameters of 12 modes, identified as physical meaningful. Closely spaced modes are observed around 8.1 Hz and also around 14.05 Hz. The mode shapes for the first 5 modes are illustrated in Figure 3.31. In particular, it is observed that the modes at 8.087 Hz and 8.175 are very similar; for the mode at 8.087 Hz the wings move however out of phase, therefore representing an anti-symmetric mode, as compared to the mode at 8.175 Hz that represents a symmetric mode with the wings moving in phase. The MAC values calculated between the first five modes are shown in Table 3.64 and also indicate a good separation of both modes at 8.1 Hz.

<table>
<thead>
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<th>RES</th>
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<td>6Z+</td>
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Table 3.63 Estimated Modal Parameters, Analysis Using References 6Z and 61Z

339
a. 4.981 Hz Mode Shape

b. 5.836 Hz Mode Shape
c. 7.739 Hz Mode Shape

d. 8.087 Hz Mode Shape
Figure 3.31 Mode Shapes for First Five Modes, Analysis Using References 6Z and 61Z

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 1.0000 & & & & \\
2 & 0.0043 & 1.0000 & & & \\
3 & 0.9731 & 0.0046 & 1.0000 & & \\
4 & 0.1452 & 0.0058 & 0.1559 & 1.0000 & \\
5 & 0.0793 & 0.0029 & 0.0549 & 0.1957 & 1.0000 \\
\end{array}
\]

Table 3.64 Modal Assurance Criterion Values for First Five Modes, Analysis Using References 6Z and 61Z

All results compared well with the results of alternative analysis procedures [113]. Finally, Figure 3.32 shows some comparisons between experimental and synthesized frequency response functions. The response locations, one on a tip of the tail wing, one on a control surface of the wing, are also indicated in Figure 3.27. It can be concluded that all the important modes are well identified.
a. (61Z, 61Z)

b. (61Z, 76Z)
Figure 3.32 Comparison of Experimental and Synthesized Frequency Response Functions, Analysis Using References 62 and 61Z
CHAPTER 4 CONCLUSIONS AND RECOMMENDATIONS

4.1 Conclusions

In search for better procedures which identify an experimental modal model from measurement data, some new analysis methods, designated direct parameter model identification methods, have been developed. These methods are designed to identify an experimental modal model by modelling measurement data directly using the constitutive differential equations. Improved experimental modal models can be identified by analyzing multiple input multiple output data, simultaneously. This enables the identification of experimental modal models with highly coupled and pseudo-repeated modes. In applying the developed methods one also has the option of using frequency response functions, or impulse response functions, free decay data or forced response data directly. Sampled force input and response sequences can be processed
directly in the time domain, avoiding therefore any truncation errors introduced by a finite discrete transform between time and frequency domain.

In the first chapter of this dissertation, direct parameter model identification methods were put in perspective with the methods that are representative of current technology. Almost all of these methods identify an experimental modal model by modelling the response in terms of the characteristic solutions of the constitutive differential equations. The characteristics of some existing direct parameter model identification methods were reviewed. Modal parameter expansions were extended to include the case of repeated modes that have a full complement of independent mode shapes.

Direct parameter model identification methods, applicable for identifying an experimental modal model from simultaneous analysis of multiple input multiple output data, were developed in Chapter 2. To make the methods general, both the situation where response measurements are available at fewer locations than expected modes and the opposite situation with response measurements at many more locations than expected modes were addressed, and solutions presented.

Frequency domain and time domain implementations were developed. The frequency domain methods were shown to have certain advantages, in particular to accommodate for
residual effects. They are however outweighed by some intrinsic peculiarities, such as poor numerical conditioning. On the other hand, the time domain methods are generally well conditioned, and fast solution techniques were applicable when the data consist of impulse response functions or forced response data with stationary random force inputs, both cases of considerable practical importance.

The developed direct parameter model identification methods were shown to have the theoretical capability to identify repeated modes with a full complement of independent mode shapes, when multiple input response data are analyzed simultaneously. Practically, this translates into a capability for identifying highly coupled, pseudo-repeated modes. It was also shown that some advanced, currently used, time domain methods, such as the Least Squares Complex Exponential method, the Ibrahim Time Domain method and the Polyreference method, can be treated as a subclass of the direct parameter model identification methods developed in this dissertation.

The discussion of applications in Chapter 3 has focused on time domain direct parameter model identification methods. Numerous analytical test cases were discussed to demonstrate the various characteristics of the developed direct parameter model identification methods, in particular the capability to obtain improved experimental modal models by
simultaneous analysis of multiple input data and direct time domain processing. The applicability on experimental data has been demonstrated using impulse response functions for several reference locations of a circular plate structure and an aircraft structure, both structures exhibiting pseudo-repeated modes.

4.2 Recommendations

The applicability of the frequency domain direct parameter model identification methods for the analysis of data that are measured in the frequency domain directly, for example using multiple input step sine force inputs, should be examined. The identification method allows for a non-uniform spacing between the different frequencies and for inclusion of residual effects, and is therefore well suited for such an analysis. Different implementations, possibly using multivariate orthogonal polynomials, might be considered to improve the numerical characteristics.

The applicability of the developed time domain direct parameter model identification methods on experimental data is so far only examined using impulse response functions as
data. The initial experiments have demonstrated that an improved modal model can be identified, in particular that highly coupled and pseudo-repeated modes can be resolved by simultaneous analysis of multiple input data. Further experiments are however required to delineate the value and limitations of these methods, especially for the direct processing of sampled force input and response sequences.

Finally, the possibility to analyze simultaneously, multiple input multiple output measurement data, raises more stringent requirements of consistency. All data that are analyzed simultaneously should describe the same time invariant structure. The development of direct parameter model identification methods benefits from, and should be paralleled by, the development of affordable multichannel data-acquisition hardware and software technology to shorten the overall measurement time and improve on the consistency of the data.
APPENDIX A: LEAST SQUARES COMPLEX EXPONENTIAL MODAL MODEL IDENTIFICATION METHODS

As was mentioned in Section 1.1, most time domain modal model identification methods use explicitly the representation of the impulse response or free decay response as a weighted superposition of modal resonators, i.e. damped complex exponentials. Among such algorithms are the Ibrahim Time Domain method [30-38] and the Least Squares Complex Exponential method [39,40]. Both methods use the impulse response or free decay response at a number of response locations, for one reference location or one set of initial condition, to identify a global estimate of pole values. The Ibrahim Time Domain method also identifies a global estimate for mode shapes. Since either method uses response data for only a single reference location or initial condition, a different set of mode shapes is potentially identified for every reference location or set of initial conditions. Recently, Vold formulated a complex exponential estimation algorithm, designated the Polyreference method [41-43], that uses response data for several reference locations or sets of initial conditions simultaneously to identify a global estimate of pole values and a unique set of mode shapes.
This method will be derived and discussed using impulse responses as data. The relation between the Polyreference method and the other two methods will be clarified. As an interesting result of these interrelations, an extension of the Ibrahim Time Domain method follows that uses, simultaneously, impulse responses or free decay responses for several force input locations or initial conditions. As was discussed in Section 2.3.2.4, these time domain modal model identification methods can also be considered as pertaining to the class of direct parameter model identification methods, that estimate separately the homogeneous and non-homogeneous part of the model.

A.1 Partial Fraction Expansion for the Impulse Response

Let $H(t)$ be a matrix of dimension $(N_o, N_i)$ with the impulse responses between $N_o$ response locations and $N_i$ force input or reference locations. A modal parameter expansion of $H(t)$ assuming $N_m$ modes is given by,

$$\sum_{i=1}^{2N_m} R_i e^{\lambda_i t}$$

(A1)  \[ H(t) = \sum_{i=1}^{2N_m} R_i e^{\lambda_i t} \]

The number of assumed modes, $N_m$, should at least equal the number of expected structure modes, $N$. All modes are counted according to their multiplicity. The pole value of mode $i$ is
represented by $\lambda_i$, and $R_i$ is a matrix of dimension $(N_o, N_i)$ with the corresponding residues. In Section 1.3.3 it was demonstrated that the residue matrices can be expressed in factored form as follows,

(A2) \[ R_i = \{V\}_i [L]_i \]

$\{V\}_i$ represents the mode shape coefficients and $[L]_i$ the modal participation factors for mode $i$. $\{V\}_i$ is a column with $N_o$ elements, and $[L]_i$ is a row of $N_i$ elements. The modal participation factors basically describe how mode $i$ is excited from the different input locations, or for the different sets of initial conditions.

Substituting $R_i$ from Equation (A2) in Equation (A1) yields,

(A3) \[ H(t) = Ve^{\Lambda t L} \]

$V$ is a matrix of dimension $(N_o, 2N_m)$ with in columns the mode shapes, $L$ is a matrix of dimension $(2N_m, N_i)$ with in rows the modal participation factors, and $e^{\Lambda t}$ is a diagonal matrix of dimension $(2N_m, 2N_m)$ with as diagonal elements $e^{\lambda_i t}$.

Let $H(t)$ be sampled with equally spaced time increment $\Delta t$, from time zero on. Substituting $t$ by $k\Delta t$ in Equation (A3),

(A4) \[ H(k\Delta t) = Ve^{\Delta (k\Delta t) L} , \ k \geq 0 \]
Or, using a condensed notation,

(A5) \( H_k = V x^k L \), \( Z = e^{\Lambda t} \), \( k \geq 0 \)

Note that the latter equation is non-linear in the modal parameters. Using a multivariate form of Prony's algorithm [29], it is however possible to derive a homogeneous finite difference equation for \( H_k \) or \( H_k^t \). The Polyreference method uses a higher order finite difference equation with matrix coefficients of dimension \((N_1, N_1)\) for \( H_k^t \), having \( \sum_{i=1}^{N_1} e^{\lambda_i t} i^k \Delta t \) as characteristic solutions. As a special case, for impulse responses corresponding to a single reference location, this finite difference equation simplifies to an equation with scalar coefficients, as used in the Least Squares Complex Exponential method. The Ibrahim Time Domain method uses a first order finite difference equation with matrix coefficient of dimension \((2N_m, 2N_m)\) for \( H_k \), having \( \{V\} i e^{\lambda_i t} i^k \Delta t \) as characteristic solutions. The original formulation of the Ibrahim Time Domain method does not include the capability for analyzing impulse responses for several references simultaneously.
A.2 Polyreference Method [41-43]

A finite difference equation for $H_k^t$ is derived first. Consider the $p$ equations that follow from shifting Equation (A5) $p$ times one time increment $\Delta t$ and let $p$ be such that,

(A6) \[ pN_i = 2N_m \geq 2N, \quad p \text{ a positive integer} \]

All such equations can be put together in one matrix equation,

(A7) \[ [H_k \, H_{k-1} \, \ldots \, H_{k-p}] = V[z^{kL} \, z^{k-1L} \, \ldots \, z^{k-pL}] \]

To avoid adding trivial equations, $k$ should be selected as,

(A8) \[ k \geq p \]

The matrix,

(A9) \[ [z^{kL} \, z^{k-1L} \, \ldots \, z^{k-pL}] \]

with $p$ from Equation (A6), has $N_i$ more columns than rows. Hence this matrix describes a linear transformation with null space of dimension at least equal to $N_i$. Therefore, non-trivial matrices $A_1, \ldots, A_p$, all of dimension $(N_i, N_i)$, can be found so that,
\[(A10) \quad [x^k_L \ x^{k-1}_L \ \ldots \ x^{k-p}_L] \begin{bmatrix} I \\ -A_1^t \\ \vdots \\ -A_2^t \end{bmatrix} = 0\]

From Equation \((A7)\) then also,

\[(A11) \quad [H_k \ H_{k-1} \ \ldots \ H_{k-p}] \begin{bmatrix} I \\ -A_1^t \\ \vdots \\ -A_p^t \end{bmatrix} = 0\]

\[(A12) \quad H_k - H_{k-1}A_1^t - \ldots - H_{k-p}A_p^t = 0\]

Or, after transposing both sides,

\[(A13) \quad H_k^t - A_1H_{k-1}^t - \ldots - A_pH_{k-p}^t = 0\]

This equation represents a finite difference equation with matrix coefficients of dimension \((N_b,N_e)\) for \(R_k^t\) \([41-43]\). As a special case, using data for only one reference location, it represents a finite difference equation with scalar coefficients \([39,40]\). To estimate the matrix coefficients, let \(k\) cover the following range,

\[(A14) \quad k = N_b \ \ldots \ N_e\]

\[N_b > p\]

All equations like Equation \((A12)\), with \(k\) in the range specified above, can succinctly be written as,
\[
\begin{bmatrix}
H_{N_b-1} & \cdots & H_{N_b-p} \\
H_{N_b} & \cdots & H_{N_b-p+1} \\
\vdots & \vdots & \vdots \\
H_{N_e-1} & \cdots & H_{N_e-p}
\end{bmatrix}
\begin{bmatrix}
A_1^t \\
\vdots \\
A_p^t
\end{bmatrix} =
\begin{bmatrix}
H_{N_b} \\
H_{N_b+1} \\
\vdots \\
H_{N_e}
\end{bmatrix}
\]

This equation is of the general form,

\[(A16) \quad AX = B\]

Using the normal equations method, a least squares estimate for \(X\) is found from,

\[(A17) \quad A^TXA = A^TB\]
\[(A18) \quad PX = D\]

with,

\[
P = \begin{bmatrix}
R_{H^tH}(0) & R_{H^tH}(-1) & \cdots & R_{H^tH}(-p+1) \\
R_{H^tH}(-1) & R_{H^tH}(-2) & \cdots & R_{H^tH}(-p+2) \\
\vdots & \vdots & \ddots & \vdots \\
R_{H^tH}(0) & R_{H^tH}(-p) & \cdots & R_{H^tH}(0)
\end{bmatrix}_{N_e \times N_b}
\]

symmetric

\[
D = \begin{bmatrix}
R_{H^tH}(1) & R_{H^tH}(1) & \cdots & R_{H^tH}(1) \\
R_{H^tH}(2) & R_{H^tH}(2) & \cdots & R_{H^tH}(2) \\
\vdots & \vdots & \ddots & \vdots \\
R_{H^tH}(p) & R_{H^tH}(p) & \cdots & R_{H^tH}(p)
\end{bmatrix}_{N_e \times N_b}
\]

\[
X = \begin{bmatrix}
A_1^t \\
A_2^t \\
\vdots \\
A_p^t
\end{bmatrix}
\]

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\[ R_{H}^{\top}H_{l}(i)_{a}^{b} = \sum_{k=a}^{b} H_{k}^{t}H_{k+i} \]

From Equation (A10) follows that \( \left[ L_{i} + te^{\lambda_{i}k\Delta t} \right] \) can be identified with a characteristic solution of the finite difference equation for \( H_{k}^{t} \), Equation (A13). In particular, consider row \( i \) of Equation (A10). that, after transposition and factoring out a common factor \( z_{i}^{k-p} \), can be expressed as,

\[(A19) \left[ z_{i}^{P}I - z_{i}^{P-1}A_{1} - \ldots - A_{p} \right] L_{i}^{t} = 0 \quad , \quad z_{i} = e^{\lambda_{i}\Delta t} \]

The finite difference equation, Equation (A13), has \( pN_{i} \) characteristic solutions. To have all poles in complex conjugate pairs, \( pN_{i} \) should be even; unless \( N_{i} \) is even, this requires \( p \) to be even. This can be made explicit, by changing Equation (A6) to Equation (A20) below,

\[(A20) \quad p'N_{i} = N_{m} \quad , \quad p' \quad \text{a positive integer} \]

With \( p' \) defined from above equation, Equation (A13) becomes,

\[(A21) \quad H_{k}^{t} - A_{1}H_{k-1}^{t} - \ldots - A_{2p}H_{k-2p}^{t} = 0 \]

The latter equation conforms to Equation (2.3.79).

The actual calculation of the pole values and modal participation factors proceeds as outlined by Equation (2.1.110) and (2.1.111).

Having an estimate for \( K \) and \( L \), an estimate for the mode
shape coefficients remains to be found. Rewrite Equation (A5) in transposed form as,

(A22) \( H_k^t = \hat{L}^t \hat{x}^k v^t \)

In this expression, \( \hat{L}, \hat{x} \) and \( V \) may represent modal parameters for all \( 2N_m \) modes identified from the finite difference equation for \( H_k^t \), or for a selected number of these modes. Let \( k \) cover the range,

(A23) \[
k = N_b \ldots N_e \\
N_b > 0
\]

Using the normal equations method, a least squares estimate for the mode shape coefficients follows from,

(A24) \[ \left[ \sum_{k=N_b}^{N_e} \hat{x}^k \hat{L}^t \hat{x}^k \right] \hat{v}^t = \left[ \sum_{k=N_b}^{N_e} \hat{x}^k \hat{L}^t H_k^t \right] \]

A.3 Ibrahim Time Domain Method [30-38] and Generalization

Instead of deriving a finite difference equation for \( H_k^t \), having \( [L]_i e^{i k \Delta t} \) as characteristic solutions, alternatively a finite difference equation for \( H_k \) can be derived, having \( [V]_i e^{i k \Delta t} \) as characteristic solutions. Consider the \( p \) equations that follow from shifting Equation (A5) \( p \) times one time increment \( \Delta t \), and let \( p \) be such that,
\[ (A25) \quad pN_0 = 2N_m > 2N \]

All such equations can be put together as follows,

\[
\begin{bmatrix}
H_k \\
H_{k-1} \\
\vdots \\
H_{k-p}
\end{bmatrix}
= \begin{bmatrix}
Vx^k \\
Vx^{k-1} \\
\vdots \\
Vx^{k-p}
\end{bmatrix} \cdot L
\]

\[ (A26) \]

Again, to avoid adding trivial equations \( k \) should be selected from,

\[ (A27) \quad k > p \]

The matrix,

\[
\begin{bmatrix}
Vx^k \\
Vx^{k-1} \\
\vdots \\
Vx^{k-p}
\end{bmatrix}
\]

\[ (A28) \]

with \( p \) defined from Equation (A25), has \( N_0 \) more rows than columns. Hence, non-trivial matrices \( A_1, \ldots, A_p \), all of dimension \( (N_0, N_0) \), can be found so that,

\[ (A29) \quad \begin{bmatrix} I & -A_1 & \ldots & -A_p \end{bmatrix} \begin{bmatrix}
Vx^k \\
Vx^{k-1} \\
\vdots \\
Vx^{k-p}
\end{bmatrix} = 0 \]
and therefore, from Equation (A26),

\[(A30) \quad [ I -A_1 \ldots -A_p ] \begin{bmatrix} H_k \\ H_{k-1} \\ \vdots \\ H_{k-p} \end{bmatrix} = 0 \]

\[(A31) \quad H_k - A_1H_{k-1} - \ldots - A_pH_{k-p} = 0 \]

This latter equation represents a finite difference equations for $H_k$. Solving for a least squares estimate of the coefficients $A_1, \ldots, A_p$ in Equation (A31) proceeds analogous as outlined for finding the least squares estimate of the coefficients in Equation (A13).

Equation (A31) can be written in linearized form, by adding the following identities.

\[(A32) \quad H_{k-i} = H_{k-i}^i, \quad i = 1 \ldots p-1 \]

Equation (A31) and all equations like Equation (A32) can succinctly be written as,

\[(A33) \quad \ddot{H}_k - A\ddot{H}_{k-1} = 0 \]
with,

\[
\mathbf{H}_k = \begin{bmatrix}
H_k \\
H_{k-1} \\
\vdots \\
H_{k-p+1}
\end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix}
A_1 & A_2 & \cdots & A_p \\
I & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & \cdots & I & 0
\end{bmatrix}
\]

Equation (A33) is, for impulse responses relative to one reference location, the fundamental equation for the Ibrahim Time Domain method [30-38]. Apparently though, the form is more general and valid for impulse responses for multiple reference locations. Solving for a least squares estimate of the matrix coefficient \( \mathbf{A} \) in Equation (A33) proceeds analogous as outlined for Equation (A13).

Equation (A33) can alternatively be derived as follows. Using modal parameters, \( \mathbf{H}_k \), defined above, can be expressed as,

\[
(A34) \quad \mathbf{H}_k = \mathbf{V} \mathbf{X}^k \mathbf{L}
\]

In this equation, \( \mathbf{V} \) is defined by,

\[
(A35) \quad \mathbf{V} = \begin{bmatrix}
\mathbf{V} \\
\mathbf{V} \mathbf{X}^{-1} \\
\vdots \\
\mathbf{V} \mathbf{X}^{-p+1}
\end{bmatrix}
\]

Shifting Equation (A34) once one time increment \( \Delta t \), yields,
(A36) \[ H_{k-1} = \tilde{\nu}x^{k-1}L \]

Equation (A34) and (A36) can succinctly be written as,

(A37) \[
\begin{bmatrix}
H_k \\
H_{k-1}
\end{bmatrix}
= \begin{bmatrix}
\tilde{\nu}x^k \\
\tilde{\nu}x^{k-1}
\end{bmatrix} L
\]

The matrix,

(A38) \[
\begin{bmatrix}
\tilde{\nu}x^k \\
\tilde{\nu}x^{k-1}
\end{bmatrix}
\]

has \(2N_m\) more rows than columns, and therefore a matrix \(A\), of dimension \((2N_m, 2N_m)\), exists such that,

(A39) \[ \tilde{\nu}x^k - A\tilde{\nu}x^{k-1} = 0 \]

With Equation (A37), this latter equation implies again Equation (A33).

From Equation (A29) follows that \(\{V\}_i e^{\lambda_i k \Delta t}\) is a characteristic solution of the finite difference equation for \(H_k\), Equation (A31). In particular,

(A40) \[
\begin{bmatrix}
z_i^P I - z_i^{p-1}A_1 - \ldots - A_p
\end{bmatrix}\{V\}_i = 0 \quad z_i = e^{\lambda_i \Delta t}
\]

It follows similarly from Equation (A39) that \(\{\tilde{V}\}_i e^{\lambda_i \Delta t}\) is a characteristic solution of the finite difference equation for \(\tilde{H}_k\), Equation (A33). In particular,

(A41) \[
\begin{bmatrix}
z_i I - A
\end{bmatrix}\{\tilde{V}\}_i \quad z_i = e^{\lambda_i \Delta t}
\]

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As mentioned before, to have all pole values occur in complex conjugate pairs, \( pN_0 \) should be even. This requires in general \( p \) to be even, which can be made explicit by changing Equation (A25) to Equation (A42) below,

\[
(A42) \quad p'N_0 = N_m \geq N, \quad p' \text{ is a positive integer}
\]

With \( p' \) defined by the above equation, Equation (A31) becomes,

\[
(A43) \quad H_k - A_1H_{k-1} - \cdots - A_{2p'}H_{k-2p'} = 0
\]

The latter equation conforms to Equation (2.3.54).

The actual calculation of pole values and mode shapes proceeds as outlined by Equation (2.1.102) and (2.1.103). Note that the mode shapes \( \{\tilde{v}\}_i \), calculated from an estimate for the matrix \( A \) in Equation (A33), do not necessarily have the structure set forth by Equation (A35). A verification of how well the mode shapes match this structure is set forth in the Ibrahim Time Domain method [33] as a criterion to distinguish structure modes from computational modes.

Having an estimate for the pole values and mode shapes, an estimate for the modal participation factors can be obtained, using equations similar to Equation (A22) through Equation (A24).
APPENDIX B  RECURSIVE SOLUTION OF THE NORMAL EQUATIONS
FOR THE CORRELATION CASE

In this appendix, a recursive solution technique is dis-
cussed for a set of simultaneous linear equations with
coefficient matrix in the form of a symmetric block Toeplitz
matrix, like for example Equation (2.3.50). With minor
changes, this technique can also be applied when the matrix
is in the form of a non-symmetric block Toeplitz matrix.
Such equations also occur when calculating, in a least
squares sense, the coefficients of a multichannel filter
([104], Chapter 6). Succinctly, such an equation can be
represented as,

\[
\begin{bmatrix}
  R_0 & R_{-1} & \cdots & R_{-p} \\
  R_1 & R_0 & \cdots & R_{-p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  R_p & R_{p-1} & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
  F_0 \\
  F_1 \\
  \vdots \\
  F_p
\end{bmatrix}
= 
\begin{bmatrix}
  G_0 \\
  G_1 \\
  \vdots \\
  G_p
\end{bmatrix}
\]  

(B1)

with,

\[
R_j = R_{-j}^t, \quad j = 1 \ldots p
\]  

(B2)

The matrices \(R_0, R_1, \ldots, R_{-p}\) all assumed of dimension
\((n,n)\), may represent the autocorrelation coefficients of some
n dimensional input sequence, the matrices $G_0$, $G_1$, ..., $G_p$, all assumed of dimension $(n,m)$, may represent the cross correlation coefficients between the n dimensional input sequence and some m dimensional output sequence. Then $F_0$, $F_1$, ..., $F_p$, all of dimension $(n,m)$, represent the coefficients of a multichannel filter of order p.

The recursive solution method, developed by Wiggins and Robinson ([101], [104], pp. 242-246), will be described.

Let \{ $I$, $A_1,k$, ..., $A_{k,k}$ \}, \{ $I$, $B_1,k$, ..., $B_{k,k}$ \} and \{ $F_0,k$, $F_1,k$, ..., $F_{k,k}$ \} represent three sequences of length $k$, having the following properties.

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} \\
R_{-1} & R_0 & \cdots & R_{-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k} & R_{-k+1} & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
I \\
A_{1,k} \\
A_{k,k}
\end{bmatrix}
= 
\begin{bmatrix}
V_{A,k} \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} \\
R_{-1} & R_0 & \cdots & R_{-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k} & R_{-k+1} & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
B_{k,k} \\
B_{k-1,k} \\
\vdots \\
I
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
V_{B,k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} \\
R_{-1} & R_0 & \cdots & R_{-k+1} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k} & R_{-k+1} & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
F_{0,k} \\
F_{1,k} \\
\vdots \\
F_{k,k}
\end{bmatrix}
= 
\begin{bmatrix}
G_0 \\
G_1 \\
\vdots \\
G_k
\end{bmatrix}
\]
Recursive relations will be developed to update above sequences to \( \{ I, A_{1,k+1}, \ldots, A_{k+1,k+1} \} \), \( \{ I, B_{1,k+1}, \ldots, B_{k+1,k+1} \} \) and \( \{ F_{0,k+1}, F_{1,k+1}, \ldots, F_{k+1,k+1} \} \), with the properties that,

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k-1}^t & R_{-k}^t & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
I \\
A_{1,k+1} \\
\vdots \\
A_{k+1,k+1}
\end{bmatrix}
= 
\begin{bmatrix}
V_{A,k+1} \\
0 \\
\vdots \\
0
\end{bmatrix}
\]  
(B6)

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k-1}^t & R_{-k}^t & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
B_{k+1,k+1} \\
B_{k,k+1} \\
\vdots \\
I
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
B_{k,k+1} \\
\vdots \\
V_{B,k+1}
\end{bmatrix}
\]  
(B7)

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k} \\
\vdots & \vdots & \ddots & \vdots \\
R_{-k-1}^t & R_{-k}^t & \cdots & R_0
\end{bmatrix}
\begin{bmatrix}
F_{0,k+1} \\
F_{1,k+1} \\
\vdots \\
F_{k+1,k+1}
\end{bmatrix}
= 
\begin{bmatrix}
G_0 \\
G_1 \\
\vdots \\
G_{k+1}
\end{bmatrix}
\]  
(B8)

When this recursion is applied, for \( k = 0 \ldots (p-1) \), then finally the sequence \( \{ F_{0,p}^*, F_{1,p}^*, \ldots, F_{p,p}^* \} \) will determine the solution for Equation (B1). The sequence \( \{ I, A_{1,k}^*, \ldots, A_{k,k}^* \} \) is also known as the prediction error sequence, the sequence \( \{ I, B_{1,k}^*, \ldots, B_{k,k}^* \} \) as the hindsight error sequence ([104], pp. 241-246). The matrices \( V_{A,k} \) and \( V_{B,k} \) represent the respective variances.
To derive such recursive relations, consider,

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k+1} & R_{-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{-k-1}^t & R_{-k}^t & \cdots & R_{-1}^t & R_0
\end{bmatrix}
\begin{bmatrix}
I \\
A_{1,k} \\
A_{2,k} \\
\vdots \\
A_{k,k}
\end{bmatrix}
= 
\begin{bmatrix}
V_{A,k} \\
0 \\
0 \\
\vdots \\
U_{A,k}
\end{bmatrix}
\]

with,

\[(B10) \quad U_{A,k} = R_{-k-1}^t + R_{-k}^t A_{1,k} + \cdots + R_{-1}^t A_{k,k}\]

And also,

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k+1} & R_{-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{-k-1}^t & R_{-k}^t & \cdots & R_{-1}^t & R_0
\end{bmatrix}
\begin{bmatrix}
0 \\
B_{k,k} \\
\vdots \\
B_{1,k} \\
I
\end{bmatrix}
= 
\begin{bmatrix}
U_{B,k} \\
0 \\
0 \\
\vdots \\
V_{B,k}
\end{bmatrix}
\]

with,

\[(B12) \quad U_{B,k} = R_{-k-1} + R_{-k} B_{1,k} + \cdots + R_{-1} B_{k,k}\]

It can be proved that \(U_{B,k}\) equals \(U_{A,k}\) ([104], pp. 242), and use will be made of this property in the proceeding. The matrices \(V_{A,k}\) and \(V_{B,k}\) are also not evaluated from above equations, but can be calculated recursively.
Consider two matrices $C_{A,k+1}$ and $C_{B,k+1}$ with the property,

(B13) $C_{A,k+1} = U_{A,k} V_{B,k}^{-1}$

(B14) $C_{B,k+1} = U_{A,k}^t V_{A,k}^{-1}$

Subtracting from Equation (B9). Equation (B11) multiplied with $C_{A,k+1}$, defined by Equation (B13), yields,

\[
\begin{bmatrix}
    R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
    R_{-1} & R_0 & \cdots & R_{-k+1} & R_{-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    R_{-k-1} & R_{-k} & \cdots & R_{-1} & R_0 \\
\end{bmatrix}
\begin{bmatrix}
    I \\
    A_{1,k} - C_{A,k+1} B_{k,k} \\
    \vdots \\
    A_{k,k} - C_{A,k+1} B_{1,k} \\
    - C_{A,k+1} \\
\end{bmatrix}
\]

Subtracting from Equation (B11). Equation (B9) multiplied with $C_{B,k+1}$, defined by Equation (B14), yields

\[
\begin{bmatrix}
    V_{A,k} - C_{A,k+1} U_{A,k}^t \\
    0 \\
    \vdots \\
    \vdots \\
    0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
    R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
    R_{-1} & R_0 & \cdots & R_{-k+1} & R_{-k} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    R_{-k-1} & R_{-k} & \cdots & R_{-1} & R_0 \\
\end{bmatrix}
\begin{bmatrix}
    - C_{B,k+1} \\
    B_{k,k} - C_{B,k+1} A_{1,k} \\
    \vdots \\
    B_{1,k} - C_{B,k+1} A_{k,k} \\
    I \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 \\
\cdot \\
\cdot \\
V_{B,k} - C_{B,k+1} U_{A,k}
\end{bmatrix}
\]

The following recursive relations result from above equations,

(B17) \( V_{A,k+1} = V_{A,k} - C_{A,k+1} U_{A,k} \)

(B18) \( V_{B,k+1} = V_{B,k} - C_{B,k+1} U_{A,k} \)

(B19) \( A_{k+1,k+1} = -C_{A,k+1} \)

(B20) \( B_{k+1,k+1} = -C_{B,k+1} \)

And, for \( j = 1 \ldots k \), using \( U_{A,k} \) as a dummy memory location,

(B21) \( U_{A,k} = A_{j,k} \)

(B22) \( A_{j,k+1} = A_{j,k} - C_{A,k+1} B_{k-j+1,k} \)

(B23) \( B_{k-j+1,k} = B_{k-j+1,k} - C_{B,k+1} U_{A,k} \)

Note that \( C_{A,k+1} \) and \( C_{B,k+1} \) can occupy the same memory locations as \( A_{k+1,k+1} \) and \( B_{k+1,k+1} \) respectively. The updated prediction and hindsight error sequences may also occupy the same memory locations as the previous sequences.

Next recursive relations are derived for the sequence \( \{F_{0,k}, F_{1,k}, \ldots, F_{k,k}\} \). Consider,
\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k+1} & R_{-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{-k}^t & R_{-k-1} & \cdots & R_{-1}^t & R_0 \\
\end{bmatrix}
\begin{bmatrix}
F_{0,k} \\
F_{k,k+1} \\
0 \\
\end{bmatrix}
= 
\begin{bmatrix}
G_0 \\
G_k \\
U_{F,k+1} \\
\end{bmatrix}
\]

With,

\[(B25) \quad U_{F,k+1} = R_{-k-1}^t F_{0,k} + \cdots + R_{-1}^t F_{k,k} \]

Using the updated sequence \{I, B_1^{k+1}, \cdots, B_{k+1}^{k+1}\} and \(V_{B,k+1}\), also,

\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
R_{-1}^t & R_0 & \cdots & R_{-k+1} & R_{-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{-k}^t & R_{-k-1} & \cdots & R_{-1}^t & R_0 \\
\end{bmatrix}
\begin{bmatrix}
B_{k+1,k+1} \\
B_{k,k+1} \\
I \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
. \\
. \\
V_{B,k+1} \\
\end{bmatrix}
\]

Consider the matrix \(C_{F,k+1}\), defined as,

\[(B27) \quad C_{F,k+1} = [U_{F,k+1} - G_{k+1}]V_{B,k+1}^{-1}\]

Note that the matrix \([U_{F,k+1} - G_{k+1}]\) can occupy the same memory locations as originally occupied by \(G_{k+1}\). Subtracting from Equation (B24). Equation (B26) multiplied with \(C_{F,k+1}\), defined by Equation (B27), yields,
\[
\begin{bmatrix}
R_0 & R_{-1} & \cdots & R_{-k} & R_{-k-1} \\
R_{-1} & R_0 & \cdots & R_{-k+1} & R_{-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
R_{-k+1} & R_{-k} & \cdots & R_{-1} & R_0
\end{bmatrix}
= \begin{bmatrix}
F_{0,k} - C_{F, k+1} B_{k+1, k+1} \\
F_{1,k} - C_{F, k+1} B_{k, k+1} \\
\vdots \\
F_{k,k} - C_{F, k+1} B_{1, k+1} \\
- C_{F, k+1}
\end{bmatrix}
\]

(B28)

\[
\begin{bmatrix}
G_0 \\
G_1 \\
\vdots \\
G_{k+1}
\end{bmatrix} = \begin{bmatrix}
F_{0,k} - C_{F, k+1} B_{k+1, k+1} \\
F_{1,k} - C_{F, k+1} B_{k, k+1} \\
\vdots \\
F_{k,k} - C_{F, k+1} B_{1, k+1} \\
- C_{F, k+1}
\end{bmatrix}
\]

The above equation suggests the following recursion formulae,

(B29) \( F_{k+1, k+1} = - C_{F, k+1} \)

(B30) \( F_{j, k+1} = F_{j, k} - C_{F, k+1} B_{k+1-j, k+1} \), \( j = 0 \ldots k \)

Note that the matrix \( C_{F, k+1} \) can occupy the same memory location as \( F_{k+1, k+1} \). The updated filter coefficients can also overwrite the cross correlation matrices \( G_0, G_1, \ldots, G_{k+1} \), when the latter ones may be destroyed.

The derived recursion formulae require the following initialization.
(B31) \( V_{A,0} = R_0 \)
(B32) \( V_{B,0} = R_0 \)
(B33) \( F_C = R_0^{-1}G_0 \)

The memory requirements of this recursive solution method for Equation (B1), assuming that the cross correlation matrices are destroyable, can be summarized as follows:

a. The autocorrelation matrices for the inputs, \((p+1)n^2\)

b. The cross correlation matrices between inputs and outputs, \((p+1)nm\)

c. The prediction error sequence, \(pn^2\)

d. The hindsight error sequence, \(pn^2\)

e. The variance of the prediction error sequence, \(n^2\)

f. The variance of the hindsight error sequence, \(n^2\)

g. Two work arrays of dimension \((n,n)\), one for the matrix \(U_{A,k}\) and one for the inversions in Equation (B13), (B14) and (B27), \(2n^2\).

A total of \(3(p+1)n^2 + (p+1)nm + 2n^2\) memory locations are therefore needed for the recursive solution technique, as compared to \((p+1)^2n^2 + (p+1)nm\) for a standard solution method.
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