I hereby recommend that the thesis prepared under my supervision by Randall J Allemang entitled Investigation of Some Multiple Input/Output Frequency Response Function Experimental Modal Analysis Techniques be accepted as fulfilling this part of the requirements for the degree of Doctor of Philosophy

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INVESTIGATION OF SOME
MULTIPLE INPUT/OUTPUT
FREQUENCY RESPONSE FUNCTION
EXPERIMENTAL MODAL ANALYSIS
TECHNIQUES

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ABSTRACT

Experimental modal analysis has become an increasingly important engineering tool during the past forty years in the aerospace, automotive, and machine tool industries. The research represented by this dissertation is concerned with developing methods which will reduce the variance of the estimates of the modal parameters determined by the frequency response function experimental modal analysis method. The primary thrust of this research is to utilize the redundant information contained in the frequency response function matrix in order to formulate the best least squares estimate of the modal parameter as well as providing additional confidence factors related to the estimation process.

Several new concepts, utilizing the redundant information in the frequency response function matrix, were formulated during this research. The concept of enhanced frequency response function is developed to use a row or column of the frequency response function matrix in order to improve frequency and damping estimates. The concepts of modal scale factor and modal assurance criterion are developed to evaluate and utilize redundant modal vector estimates in order to improve the modal vector estimates.
Finally, in order to obtain multiple columns of the frequency response function matrix simultaneously, the multiple input frequency response function method is investigated. The primary reason for this new approach to the estimation of frequency response functions is to improve the consistency between elements of the frequency response function matrix by distributing the energy of excitation throughout the system. This tends to desensitize the nonlinear effects which are a function of excitation location and response amplitude for a given level of input energy.

The concepts and techniques, evaluated in the course of this research, were demonstrated experimentally on several test objects, including a T-plate test structure, a Schweitzer sailplane, and an automotive frame.
This dissertation is dedicated to the concept that there is always time for "just one more set of data" but often no reasonable way to analyze the additional data in a rigorous way. It is my hope that the techniques developed within the bounds of this dissertation provide some reasonable procedures for utilizing the redundant frequency response function information for experimental modal analysis.

It should be noted that some of this work was made possible by grants and/or contracts with other organizations. Specifically, NASA–Langley Research Center, Instrument Research Division, has supported this work under NSG-1486 in order to further research in the general area of frequency response function experimental modal analysis. The Boeing Company indirectly supported this work by utilizing the University of Cincinnati as a subcontractor on the Air Force contract entitled "Optimum Ground Vibration Test Method". Wright-Patterson Air Force Base, Flight Dynamics Lab provided indirect support through the "Optimum Ground Vibration Test Method" contract as well as direct support in the form of the loan of the Schweitzer sailplane used in much of the experimental work. Finally, Eglin Air Force Base provided direct support of the multiple input estimation work on the ongoing
contract entitled "Dual Input Random Excitation Experimental Modal Analysis Study".

There are many individuals who have contributed to the completion of this work. The list is long and I can only hope that each one knows the extent of my appreciation.
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1.1 OVERVIEW

Experimental modal analysis has become an increasingly important engineering tool during the past forty years in the aerospace, automotive, and machine tool industries. The modal parameter estimates obtained from experimental modal analysis are being used in the direct solution of vibration and/or acoustic problems, for correlation with output from finite element programs, and for prediction of changes in system dynamics due to structural changes. In all cases, the quality of the modal parameter estimates is of major concern.

1.2 RESEARCH OBJECTIVES

The primary thrust of this research is to improve and/or verify the modal parameter estimates derived from experimental modal
analysis. While several experimental modal analysis approaches are currently in use, the single input frequency response function method appears to be the most general, proven method when a controlled force experiment can be conducted. Based upon an extensive literature review and aerospace industry interviews [1-6], any of the current experimental modal analysis methods can produce approximately the same results given unlimited access to hardware, expertise, time, and funds. The single input frequency response function method is more attractive since the general input/output concept is utilized in many areas outside of structural analysis. For this reason, more cross transfer of information concerning measurement techniques, parameter estimation algorithms, and evaluation schemes from non-related fields is possible than with the other techniques. Even though the single input frequency response function to experimental modal analysis appears to be the most attractive, the single input frequency response function method, as in the other methods, does not currently provide any methodology for utilizing redundant information obtained from independent estimates of the modal vectors.

Therefore, this research is concerned with developing methods for utilizing the redundant information contained in independent estimates of the modal vectors of a system. With respect to the single input frequency response function method of experimental modal analysis, these independent estimates of modal vectors can
be derived from the rows and/or columns of the residue matrices obtained from the frequency response function matrix. These separate estimates can be obtained through repeated application of a single input controlled force experiment. Additionally, separate estimates can result from different parameter estimation algorithms or different testing constraints. This approach parallels the effort to estimate the frequency and damping of each modal vector by utilizing all or any part of the elements of the frequency response function in a least squares process. In each case, redundant information can be utilized to provide the best least squares estimate of the modal parameter as well as providing additional confidence factors related to the estimation process. As a result of this approach, all of the redundant characteristics of the frequency response function matrix will be utilized in the calculation of the modal parameters of the system.

Finally, in order to obtain multiple columns of the frequency response function matrix simultaneously, the multiple input frequency response function method will be investigated. The primary reason for evaluating this new approach to the estimation of frequency response functions is to improve the consistency between elements of the frequency response matrix by distributing the energy of the excitation throughout the system. This will tend to desensitize the nonlinear effects which are a function of excitation location and response amplitude for a given level of
input energy.

1.3 RESEARCH APPLICATION

The ultimate goal of this research is to improve the modal parameter estimates derived from the frequency response function experimental modal analysis method. Primarily, then, the results will have direct application to the current single input approach. Secondarily, though, this research will provide the groundwork for investigation of the potential for a multiple input frequency response function experimental modal analysis method. Both of these approaches to modal parameter estimation require systematic methods for combining, evaluating, and verifying independent estimates of the same modal vector. Based upon the current trend in experimental hardware, measurement systems with a large number of parallel input channels, extensive computer memory, vector arithmetic processing, and virtually unlimited data storage are going to be readily available in the near future. This will increase the need for methodology oriented toward handling more and more of the frequency response function matrix.
1.4 RESEARCH OUTLINE

In order to obtain some perspective as to the relative merit of the frequency response function method of experimental modal analysis, a complete review of current experimental modal analysis methods was made in terms of a literature search and personal interviews. A summary of this review is included in Chapter 2. The bibliography list is included in Appendix D. This review showed that there was little difference in the theoretical basis for each method with the exception of the real mode restriction of the forced normal mode excitation method. Even though the current methods are quite similar, a number of problem areas have still not been resolved. None of these methods utilize any sort of systematic approach for evaluating independent estimates of the same modal vector but each method allows that different estimates can exist due to experimental procedure. In addition, the frequency response function method demonstrates a lack of confidence factors related to the estimation of modal vectors. A least squares evaluation technique involving a modal scale factor and a modal assurance criterion are formulated in Chapter 5 to address these problems.

The accurate measurement of the frequency response function is
fundamental to the eventual modal parameter estimation. Chapter 3 represents a summary of concepts related to frequency response function estimation. These concepts are a composite of theory and application that the author has found to be vital for accurate measurement of the frequency response function.

A method for improving the estimate of frequency and damping for a modal vector is presented in Chapter 4. This method is based upon a least squares generation of a forced response function using a set of frequency response functions and an estimate of the modal vector. If the modal vector estimate is the output of a finite element program, the forced response function can be used as a measure of correlation of the finite element model to the system being tested.

Finally, Chapter 6 reviews the theory and presents some initial experimental work concerning the multiple input estimation of frequency response functions for experimental modal analysis. The results of the work clearly indicate that, at least for certain structures, the multiple input estimation method is practical at the current stage of development. Additionally, much insight has been gained as to the problem areas and potential improvement of the current approach to multiple input estimation of frequency response functions.
1.5 DEFINITION OF TERMINOLOGY

In the discussion of experimental modal analysis, terminology can often be misinterpreted. For this reason the following terminology has been defined in order to distinguish among many subtle differences:

Eigenvalue

A non-trivial complex valued solution of the characteristic equation of a system. This is also referred to as a characteristic value. Since the characteristic equation must be known, the eigenvalue will be the result of the solution of a theoretical formulation of the system dynamics problem.

System Pole

The complex valued representation of the frequency and damping associated with a mode of vibration of the system. This can be measured experimentally or estimated theoretically. Therefore, an eigenvalue is an estimate of a system pole.
Eigenvector

A non-trivial complex valued vector associated with the solution of an eigenvalue problem. This may also be referred to as a characteristic vector or function. This solution is a result of a theoretical formulation of the system dynamics problem.

Modal Vector

A complex valued vector representing the relative motion relationship of all points on a structure at a pole of the system. The scaling of the vector has meaning only with respect to the elements of the vector. Therefore, an eigenvector is an estimate of a modal vector.

Mode Shape Vector

A complex valued vector representing the motion relationship of all points on a structure at a pole of the system. The individual elements of the mode shape vector are scaled according to a specific scheme and, thus, the mode shape vector has meaning in both a relative and absolute sense. The scaling is often performed so that the set of mode shape vectors is orthonormal or orthogonal with respect to a mass matrix. The mode shape vector is a modal vector but is not necessarily an eigenvector.
Modal Coefficient

A specific complex valued element of a modal vector or an eigenvector.

Residue

A specific complex valued coefficient in the numerator of the partial fraction expansion representation of a frequency response function or transfer function. This value is a function of the excitation location, the response location, and a scaling constant. Therefore, a residue is a modal coefficient but the converse is not true.

Residue Matrix

The residue matrix is the matrix of residues representing all input-output modal coefficient combinations for only one mode of vibration.

Modal Matrix

A matrix made up of all the modal vectors of a system arranged in increasing order of frequency as columns of the modal matrix.
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2.1 OVERVIEW

In order to evaluate and improve any approach to experimental modal analysis, the relative merits of all viable techniques must be well understood. To that goal, many articles have been written to try to compare and contrast the value of one method over another. Unfortunately, most of these comparisons have been heavily concerned with differences that are a function of specific implementations of the various techniques. These comparisons were also potentially biased by the expertise of the test engineers being restricted to only one of the areas of testing. Since each method involves very special testing awareness, this sort of analysis has limited value.

In the evaluation of experimental modal analysis methods, the differences in the theoretical approach are obviously of prime concern. Since most experimental modal analysis methods involve similar theoretical basis, the only significant areas of difference concern the concept of real versus complex modal
vectors and the explicit measurement of the input. The debate over the need to describe complex valued modes of vibration may never end. Certainly, the concept of a complex mode, since it contains a real mode as a special case, appears to be the most general case. Likewise, some experimental modal analysis methods do not require the measurement of the input. While this can be advantageous at times where the implicit nature of the input is known or assumed, it seems prudent, where the input can be measured, to do so.

Beyond the direct theoretical differences, though, there are several key evaluation considerations which may or may not be a direct function of the theory. The availability of confidence factors, the potential for implementation, and the need for operator expertise may control the ability to estimate valid modal parameters. Specifically, through the knowledge of these aspects with respect to other experimental modal analysis methods, the frequency response function method may be greatly improved.

2.2 SYSTEM ASSUMPTIONS

There are three basic assumptions concerning any structure that are made in order to perform an experimental modal analysis.
First, the structure is assumed to be linear. This means that the response of the structure to any combination of forces, simultaneously applied, is the sum of the individual responses to each of the forces acting alone. For a wide variety of structures this is a very good assumption. When a structure is linear, its behavior can be characterized by a controlled excitation experiment in which the forces applied to the structure have a form convenient for measurement and parameter estimation rather than being similar to the forces that are actually applied to the structure in its normal environment. For many important kinds of structures, however, the assumption of linearity is not valid. In these cases it is hoped that the linear model that is identified provides a reasonable approximation of the structure's behavior.

The second basic assumption is that the structure is time invariant. This essentially means that the parameters that are to be determined are constants. In general, a system which is not time invariant will have components whose mass, stiffness, or damping depend on factors that are not measured or are not included in the model. For example, some components may be temperature dependent. In this case the temperature of the component is viewed as a time varying signal, and, hence, the component has time varying characteristics. Therefore, the modal parameters that would be determined by any measurement and estimation process would depend on the time (by this temperature
dependence) that any measurements were made. If the structure that is tested is changing with time, then measurements made at the end of the test period would determine a different set of modal parameters than measurements made at the beginning of the test period. In other words, the measurements made at the two different times will be inconsistent because of the assumption of time invariance.

The third basic assumption is that the structure is observable. This means that the input-output measurements that are made contain enough information to generate an adequate behavioral model of the structure. Structures and machines which have loose components, or more generally, which have degrees of freedom of motion that are not measured, are not completely observable. Consider describing the motion of a partially filled tank of liquid when complicated sloshing of the fluid occurs. Sometimes enough measurements can be made so that the system is observable under the form chosen for the model, and sometimes no realistic amount of measurements will suffice until the model is changed.

2.3 EXCITATION ASSUMPTIONS

There are a number of techniques currently available that can be used to estimate modal characteristics from response measurements
with no measurement of the excitation. All of these techniques have one thing in common: an assumption has to be made concerning the excitation of the system. Usually, one assumes that the autospectrum (or power spectrum) of the excitation signal is sufficiently smooth over the frequency interval of interest.

In particular, the following assumptions about the excitation signal can be used:

1) The excitation is impulsive. The autospectrum of a short pulse (time duration much smaller than the period of the greatest frequency of interest) is nearly uniform, or constant in amplitude, and largely independent of the shape of the pulse.

2) The excitation is white noise. White noise has an autospectrum that is uniform over the bandwidth of the signal.

3) The excitation signal is a step. A step signal has an autospectrum that decreases in amplitude in proportion to the reciprocal of frequency. The step signal can be viewed as the integral of an impulsive signal.

4) The excitation is a Wiener-Levy signal. This signal can be treated as the integral of a white noise signal, and has an autospectrum similar to that of the step signal.
5) There is no excitation. This is called the free response or free decay situation. The structure is excited to a condition of nonzero displacement, or nonzero velocity, or both. Then the excitation is removed, and the response is measured during free decay. This kind of response can be modeled as the response of the structure to an excitation signal that is a linear combination of impulsive and step signals.

When the excitation autospectrum is uniform, the autospectrum of the response signal is proportional to the square of the modulus of the frequency response function. In this case, the poles of the response spectrum are the poles of the frequency response, which are the parameters of the system resonances. If the autospectrum is not uniform, then the excitation spectrum can be modeled as an analytic function, to a precision comparable to typical experimental error in the measurement of spectra. In this model, the excitation spectrum has poles that account for the nonuniformity of the spectrum amplitude. The response signal, therefore, can be modeled by a spectrum that contains zeros at the zeros of the excitation and the zeros of the frequency response, and contains poles at the poles of the excitation and at the poles of the frequency response. It is obviously important that the force spectrum should have no poles or zeros which coincide with poles of the frequency response.
For transient inputs, such as an impact or step relaxation, the assumption of smooth excitation spectra is generally true, but for operating inputs or inputs generated by an exciter system, care must be taken to insure the input force spectrum is smooth. This is especially true for tests performed using a hydraulic or an electro-mechanical exciter, because the system being analyzed may "load" the exciter system (the structure's impedance is so low that the desired force level cannot be achieved within the constraint of small motion), and this causes a nonuniformity in the input force spectrum. In Figure 2-1, a typical example of exciter loading is shown. The input force spectrum is shown in a frequency band around one of the structure resonances. Also shown are the response autospectrum and the measured frequency response.

To determine the characteristics of the system from the response, it is necessary that the response have the same poles as the frequency response, or that the analysis process corrects for the zeros and poles of the excitation. As can be seen from Figure 2-1, the force input spectrum does have a zero in the frequency range of interest, and as a result the pole location measured from the response spectrum does not match that of the frequency response. Presently, there is a great deal of interest in determining modal parameters from measured response data taken on operating systems (for example: turbulent flow over an airfoil, road inputs to automobiles, and environmental inputs to proposed
Figure 2-1: Typical Example of Exciter Loading
large space structures). For these cases, care must be taken not to confuse poles that are system resonances with those that exist in the output spectrum due to inputs.

In general, the poles of the response include those of the frequency response and of the input spectrum. Therefore, if the force is not measured, it is not possible without some prior knowledge about the input to determine if the poles of the response are truly system characteristics. If no poles or zeros exist in the force spectrum in the frequency range of interest, then any poles in the response in this range must be a result of the system characteristics.

2.4 FORCED NORMAL MODE EXCITATION

The forced normal mode excitation method of experimental modal analysis is the oldest approach to the estimation of dynamic structural parameters. Currently this method is still used extensively in the aerospace industry for ground vibration testing of aircraft structures. This method was originally outlined in an article by Lewis and Wrisley in 1950 [1]. Very simply stated, Lewis and Wrisley found that a number of exciters could be tuned to exactly balance the dissipative forces in a structure at certain frequencies. A typical example of this
approach is represented in Figure 2-2. When the force balance is achieved, the differential equations of motion describing the structure can be reduced to the undamped homogeneous differential equations of motion at that particular frequency. In order to more thoroughly explain this phenomena, DeVeubeke in 1956 published an article [2] which explains the theoretical basis for this testing in terms of Characteristic Phase Lag Theory. Further development of the practical application of this theory was enhanced by the concept of effective number of degrees of freedom by Traill-Nash [3]. This concept explains that the required number of exciters is a function of the effective number of degrees of freedom not the total number of degrees of freedom. The effective number of degrees of freedom is a function of modal density and damping. Finally, Asher utilized the determinant of the real part of the frequency response matrix to locate damped natural frequencies and determine the effective number of degrees of freedom [4,5].

Most of the work done since 1958 has been concerned with improvements in the implementation of the method, primarily the force appropriation [5-12]. The most advanced implementation at this time has been accomplished by the German research organization, DFVLR. This system involves nearly five hundred channels of data acquisition and co/quad analysis equipment controlled from a mini-computer. Extensive tuning criteria are utilized as well as real time animated displays of the modal
Forced Normal Mode Excitation

Figure 2-2: Forced Normal Mode Excitation
vector as well as of the out of phase response. This can be particularly useful for optimum exciter location as well as force appropriation [13].

2.4.1 THEORY

The development of the forced normal mode excitation method of experimental modal analysis begins with the matrix differential equation for an n degree of freedom system undergoing excitation at a single frequency.

\[
\begin{align*}
(2-1) \quad \begin{bmatrix} M \\ C \\ K \end{bmatrix} \ddot{x} + \begin{bmatrix} C \end{bmatrix} \dot{x} + \begin{bmatrix} K \end{bmatrix} x &= \{f\}
\end{align*}
\]

Using an excitation of the form shown, the response should be of the same form with some phase lag.

\[
(2-2) \quad \{f\} = \{F\} \sin(\omega t)
\]

\[
(2-3) \quad \{x\} = \{X\} \sin(\omega t - \phi)
\]

Ideally, then, if \(\{F\}, \omega\) and \(\phi\) are chosen properly, Equation 2-1 can be reduced to:
(2-4) \[ [M] \ddot{x} + [K] x = 0 \]

Since, under these conditions:

(2-5) \[ [C] \dot{x} = f \]

when Equations 2-2 and 2-3 are substituted into Equation 2-1, similar terms can be collected. Collecting all terms containing a \( \sin(\omega t) \) and dividing through by \( \sin(\omega t) \) gives:

(2-6) \[ \cos(\phi) \left[ [K] - w^2 [M] \right] x + w \sin(\phi) [C] x = F \]

Collecting all terms containing a \( \cos(\omega t) \) and dividing through by \( \cos(\omega t) \) gives:

(2-7) \[ -\sin(\phi) \left[ [K] - w^2 [M] \right] x - \cos(\phi) [C] x = 0 \]

These two equations are the basis of the forced normal mode excitation method. In order to understand the application of Equations 2-6 and 2-7, two cases must be evaluated. For the first case, if the \( \cos(\phi) \) is zero, the phase lag angle \( \phi \) must be 90 degrees within multiples of 180 degrees. If this is true, Equation 2-6 reduces to:

(2-8) \[ w [C] x = F \]
Equation 2-7 reduces to:

\[(2-9) \quad \left[ [K] - w^2[M] \right] \{X\} = \{0\}\]

Therefore, for the non-trivial solution of \{X\}:

\[(2-10) \quad \left| [K] - w^2[M] \right| = 0\]

Equation 2-10 is recognizable as the characteristic equation of the undamped system. Obviously, if the input vector can be adjusted according to Equation 2-8, the response vector will be the modal vector for the natural frequency.

For the second case, if the Cos(\(\phi\)) is non-zero, Equation 2-7 can be divided by a negative Cos(\(\phi\)) with the following result:

\[(2-11) \quad \left[ \Tan(\phi) \left[ [K] - w^2[M] \right] - w [C] \right] \{X\} = \{0\}\]

Again, for the non-trivial solution of \{X\}, Equation 2-11 will have roots of the following equation:

\[(2-12) \quad \left| \Tan(\phi) \left[ [K] - w^2[M] \right] - w [C] \right| = 0\]

Therefore, for any frequency of excitation, an m-th order solution of Equation 2-12 exists in terms of phase lag angle \(\phi\). This can be interpreted in the following way. At any frequency,
a forced normal mode of vibration can exist but the phase lag angle will be other than 90 degrees. Once a modal vector is tuned using a 90 degree phase lag angle criteria, the excitation frequency can be varied with no change in modal vector. This can be used as a check to determine whether the modal vector has been adequately tuned.

In addition to this potential confidence check, the excitation can be removed from the system once a modal vector is tuned. If the modal vector contains only responses due to a single mode of vibration, the exponential decay at all response positions should contain only the excitation frequency and the envelope of exponential decay should give an accurate estimate of the system damping.

The forced normal mode excitation method works well in the presence of proportional damping but theoretically does not include the concept of complex modes of vibration. Due to this theoretical limitation, the practical application of the 90 degree phase lag criteria is normally applied only to within plus or minus 10 degrees. Likewise, added difficulty is encountered in evaluating the exponential decay purity as well as force appropriation. Much work has been done on automated tuning algorithms to alleviate this. These algorithms alter excitation magnitude and phase to try to achieve 90 degree phase lag criteria under severe impedance matching situations.
Unfortunately, the location of the excitation cannot be evaluated automatically in this process.

2.5 FREQUENCY RESPONSE FUNCTION

The frequency response function method of experimental modal analysis is the most commonly used approach to the estimation of modal parameters. This method originated as a testing technique as a result of the use of frequency response functions in the forced normal mode excitation method to determine natural frequencies and effective number of degrees of freedom [14-19]. With the advent of the computer and mini-computer, the frequency response function method became a separate, viable technique [20-33].

In this method, frequency response functions are measured using excitation at a single point. This excitation may be narrow band or broadband as well as being random or deterministic. The frequency response functions are used as input to a modal parameter estimation process to determine the modal parameters. Often, a curvefitting procedure is used to evaluate the parameter estimation procedure. An orthogonality check may be made of the modal vectors to evaluate the relationship of the estimates to some mathematical model.
In order to be sure that all modal vectors have been found experimentally, a number of excitation points must be utilized, one at a time. This reduces the possibility of trying to excite the system at a node of one of the modal vectors.

2.5.1 THEORY

The frequency response function method begins with the matrix differential equation of motion description of an n degree of freedom system.

\[(2-13) \quad [M] \ddot{x} + [C] \dot{x} + [K] x = f\]

In this statement of the system equations of motion, only viscous damping appears although other damping mechanisms may be present. Potter [25] has shown that only the viscous component of these other damping mechanisms actually contribute to the dissipation of energy. Therefore \([C]\) can be defined as the equivalent viscous damping matrix for the system.

Taking the Laplace transform of Equation 2-13, assuming that all initial conditions are zero, gives:

\[(2-14) \quad \left[ [M]s^2 + [C]s + [K] \right] \{X(s)\} = \{F(s)\}\]
In order to simplify Equation 2-14 the concept of system matrix 
\([B]\) is defined.

\[
(2-15) \quad [B(s)] = \begin{bmatrix} [M]s^2 + [C]s + [K] \end{bmatrix}
\]

Therefore:

\[
(2-16) \quad [B(s)] \{X(s)\} = \{F(s)\}
\]

The transfer function matrix \([H]\) can be defined:

\[
(2-17) \quad \{X(s)\} = [H(s)] \{F(s)\}
\]

Therefore:

\[
(2-18) \quad [H(s)] = [B(s)]^{-1}
\]

Obviously the number of measurement points \(m\) cannot be greater 
than the number of elements of the excitation vector for the transfer 
function to be equal to the left handed inverse of the system matrix. The number of elements of the excitation vector can always be supplemented by zeroes to equal \(m\).

The inverse of the system matrix can be given by:
\[(2-19) \quad [H] = [B]^\dagger = \frac{[D]}{\text{DET}[B]} \]

where:
- \([D]\) = Adjoint matrix of \([B]\)
- \([D]\) = Transpose of cofactor matrix of \([B]\)
- \(\text{DET}[B]\) = Determinant of \([B]\)

Since both the adjoint matrix of \([B]\) and the determinant of \([B]\) are polynomials in \(s\), the elements of \([H]\) are rational fractions in \(s\).

Therefore, it is possible to represent any element of the transfer function matrix \([H]\) in a partial fraction form.

If it is assumed that the poles of the system are of unit multiplicity, and that the system is inherently underdamped (the poles occur in complex conjugate pairs) the transfer function matrix can be expressed as

\[(2-20) \quad [H] = \sum_{r=1}^{n} \frac{[A(r)]}{s - p(r)} + \frac{[A(r)]^*}{s - p^*(r)} \]

where:
- \(p(r)\) = System pole
- \(p(r) = \sigma(r) + j\ \text{wd}(r)\)
- \(\sigma(r)\) = Damping factor
- \(\text{wd}(r)\) = Damped natural frequency
- \(^*\) = Complex conjugate
- \([A]\) = Residue matrix
The undamped natural frequency is given by:

\[
(2-21) \quad \omega_n(r) = \sqrt{\sigma'(r)^2 + \omega_d(r)^2}
\]

The critical damping factor, \( \zeta(r) \), is given by:

\[
(2-22) \quad \zeta(r) = -\frac{\sigma'(r)}{\omega_n(r)}
\]

The modal vectors can be described in terms of the adjoint matrix of \([B]\). Using Equation 2-19 with the following identity, the basis for the modal vectors can be developed.

\[
(2-23) \quad [B] [B]^{-1} = [I]
\]

Substituting Equation 2-19 into Equation 2-23 yields:

\[
(2-24) \quad [B] [D] = \text{DET}[B] [I]
\]

Evaluating Equation 2-24 at any one of the system roots causes the determinant of \([B]\) to go to zero.

\[
(2-25) \quad [B] [D] = [0]
\]

Equation 2-25 can be rewritten using only the j-th column of the adjoint matrix \([D]\).
(2-26) \[ [B] \{D(j)\} = \{0\} \]

Equation 2-26 looks exactly like the homogeneous solution of Equation 2-16 when evaluated at any one of the system poles. Therefore, the j-th column of the adjoint matrix [A] is an estimate to within an arbitrary constant of the modal vector for that pole. Since the j-th column was chosen arbitrarily, any column of the adjoint matrix is an estimate of the modal vector of that system pole to within a constant. Finally, from Equation 2-19, it can be easily seen that the transfer function matrix is simply equal to the adjoint matrix, at any value of s, to within a constant.

If the transfer function matrix is limited to values of s having zero real part, this special case results in the frequency response matrix.

Therefore, if the elements of the frequency response matrix can be measured, each column will contain information which can be used to estimate modal vectors. Since the frequency response matrix is considered to be symmetric due to the Maxwell-Betti relations, each row will also contain the information needed to estimate modal vectors.

In order to obtain estimates of the modal vectors, the frequency response functions are used as input in a parameter estimation scheme based on Equation 2-20.
\( (2-27) \quad H(i,j) = P(i,j) + \sum_{r=1}^{n} \frac{a(i,j,r)}{s - p(r)} + \frac{\ast a(i,j,r)}{s - \ast p(r)} + Q(i,j) \)

where: 
- \( s \) = Laplace variable
- \( a(i,j,r) \) = Residue
- \( p(r) \) = System pole
- \( n \) = Number of degrees of freedom
- \( P(i,j) \) = Residual flexibility
- \( Q(i,j) \) = Residual inertia

Often Equation 2-27 is altered by an assumption of real modes, a specific damping mechanism, or known system poles. Under such assumptions, the evaluation of Equation 2-27 for the remaining modal parameters becomes much simpler.

2.6 DAMPED EXPONENTIAL RESPONSE FUNCTION

The damped exponential response function method of experimental modal analysis is one approach receiving considerable attention at the present time. The only practical application of this method is the Ibrahim time domain (ITD) method [34-38], developed to extract the modal parameters from damped exponential response function information. Digital free decay response data are
measured at various points on the structure. If response data from all the selected measurement positions cannot be obtained simultaneously because of equipment restrictions, a common position is retained between measurement groups. A recurrence matrix is created from the free decay data, and the eigenvalues of this matrix are exponential functions of the poles of the system, from which the poles are easily computed. The eigenvectors of the recurrence matrix are response residues, from which the mode shapes are determined.

The damped exponential response function method is rather straightforward in application if the necessary data acquisition hardware, computer facilities and software are available. This approach computes the poles and residues based upon a specific initial vibration condition of the structure. A number of different initial conditions can be established, analogous to the practice of using several exciter positions in ordinary single input modal surveys, until all the important modes have been excited. All the modes cannot be established from one exciter position, and likewise all the modes cannot be determined from one initial condition.

Although this technique is based upon free decay data, this procedure can be used with operating inputs if the free decay is computed from the operating inputs by using "Random-decrement" [37] or from measured autocorrelation and cross correlation
functions. Again, it should be emphasized that this can only be done if there are no poles or zeros in the input spectrum in the frequency range of interest.

2.6.1 THEORY

The damped exponential response function method as applied by Ibrahim begins with the differential equations of motion for the free response situation.

\[(2-28) \quad [M] \ddot{\{X\}} + [C] \dot{\{X\}} + [K] \{X\} = \{0\}\]

The response vector for each of the measurement points on the structure is a function of the system poles and associated modal vectors.

\[(2-29) \quad \{X\} = \{P\} \exp(\lambda t)\]

where: \(\exp(\lambda t) = e^{\lambda t}\)

For a particular element of the response vector, Equation 2-29 can be written:

\[(2-30) \quad x = \sum_{j=1}^{2m} P(j) \exp(\lambda(j) t)\]
Substituting Equation 2-29 into Equation 2-28 yields:

\[(2-31) \begin{bmatrix} [M] \lambda^2 + [C] \lambda + [K] \end{bmatrix} \{P\} = \{0\}\]

where: \( \lambda = \) Complex system pole
\( \lambda = \sigma + j \omega \)
\( \{P\} = \) Displacement vector
\( P(j) = \) Displacement component of mode \( j \) for a particular point at a particular time

Equation 2-30 can be rewritten for a specific instance in time.

\[(2-32) X(i) = \sum_{j=1}^{2n} P(j) \exp(\lambda(j) t(i))\]

where: \( X(i) = X(t(i)) \)

Data is now gathered at 2n time intervals. Equation 2-32 can be reformulated as a matrix product to reflect this as follows:

\[(2-33) \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} E \end{bmatrix}\]

where: \( m = \) Number of measurement stations
\( n = \) Number of degrees of freedom
\( E(r,c) = \exp(\lambda(r) t(c)) \)
\( r = \) Row index
c = Column index

[X] = m x 2n

[P] = m x 2n

[E] = 2n x 2n

This process is now repeated two additional times. Data is gathered for 2n time increments. Each set of data is measured at an increment of $\Delta t$ later compared to the previous set. This results in the following equations. For the first additional data set:

\begin{equation}
(2-34) \quad [Y] = [Q] [E]
\end{equation}

\begin{equation}
(2-35) \quad y(i) = x(t(i) + \Delta t)
\end{equation}

\begin{equation}
(2-36) \quad Q(i,j) = P(i,j) \exp(\lambda(i) \Delta t)
\end{equation}

For the second additional data set:

\begin{equation}
(2-37) \quad [Z] = [R] [E]
\end{equation}

\begin{equation}
(2-38) \quad Z(i) = Y(t(i) + \Delta t) = X(t(i) + 2\Delta t)
\end{equation}
Equations 2-33 and 2-34 can be combined into a single matrix equation.

\[(2-40) \quad [\phi] = [\psi] [E]\]

where:
\[
[\phi] = \begin{bmatrix} [X] \\ [Y] \end{bmatrix}
\]
\[
[\psi] = \begin{bmatrix} [P] \\ [Q] \end{bmatrix}
\]

Likewise, Equations 2-34 and 2-37 can be combined into a single matrix equation.

\[(2-41) \quad [\phi'] = [\psi'] [E]\]

where:
\[
[\phi'] = \begin{bmatrix} [Y] \\ [Z] \end{bmatrix}
\]
\[
[\psi'] = \begin{bmatrix} [Q] \\ [R] \end{bmatrix}
\]

Equations 2-40 and 2-41 can be manipulated into the form of Equation 2-42 if the inverses of \([\phi]\), \([\phi']\), \([\psi]\), and \([\psi']\) exist. The inverses of \([\psi]\) and \([\psi']\) exist, since these
matrices are functions of modal vectors (which should be linearly
independent). The inverses of \([ \phi \] and \([ \phi' \] will exist as long
as the matrix \([E] \) is non-singular. This will be the case as long
as there is not a repeated root of the system. Therefore:

\begin{equation}
(2-42) \quad [ \phi' ] [ \phi ]^{-1} [ \psi ] = [ \psi ]
\end{equation}

For corresponding columns of \([ \psi ] \) and \([ \psi' ] \):

\begin{equation}
(2-43) \quad [ \phi' ] [ \phi ]^{-1} \psi(i) = \psi'(i)
\end{equation}

But these columns are already related as shown by Equations 2-36
and 2-39.

\begin{equation}
(2-44) \quad \psi(i) = \exp( \lambda(i) \Delta t ) \psi(i)
\end{equation}

Rewriting Equation 2-43 to reflect this gives:

\begin{equation}
(2-45) \quad [ \phi' ] [ \phi ]^{-1} \psi(i) = \exp( \lambda(i) \Delta t ) \psi(i)
\end{equation}

Equation 2-45 is an eigenvalue problem whose eigenvalues and
eigenvectors are related to the estimates of the dynamic system
parameters. The system poles can be found from the eigenvalues
of Equation 2-45.
\[(2-46) \quad \sigma = \frac{1}{2 \Delta t} \ln(a^2 + b^2)\]

\[(2-47) \quad wd = \frac{1}{2 \Delta t} \arctan \left( \frac{b}{a} \right)\]

where: \(a + jb = \text{Eigenvalue of Equation 2-45}\)

Since Equation 2-47 does not result in unique solutions for the damped natural frequency, the sampling rate must be chosen to preclude unwanted frequency ranges of analysis.

The modal vectors are determined in a straightforward manner as the first \(n\) elements of the eigenvectors of \([\phi'] [\phi]^{-1}\).

An additional development with respect to this technique is the concept of modal confidence factor [34]. The modal confidence factor is a complex number calculated for each identified mode of the structure, while undergoing an exponential decay form of vibration test. The theory of the modal confidence factor is based upon the modal deflection at a particular measurement point being related to the modal deflection at that same measurement point at any time earlier or later in the free decay response. Therefore, if modal vectors are estimated from exponential decay data and two separate estimates are calculated from sets of data taken some fixed time \(\Delta t\) apart, the relationship between the
modal vector estimates will be as follows:

\[(2-49) \quad A = Q \exp(\lambda(i) \Delta t)\]

where: \(\lambda(i)\) = Estimated system pole
\(Q\) = First estimate of modal vector
\(Q'\) = Second estimate of modal vector
\(A\) = Calculated estimate of modal vector

The modal confidence factor is defined, then, as the relationship between the measured second estimate of the modal vector and the calculated estimate of the modal vector based upon the measured first estimate of the modal vector.

\[(2-49) \quad A = Q' \cdot MCF\]

Finally, the modal confidence factor can be stated as follows:

\[(2-50) \quad MCF = \begin{cases} \frac{A}{Q'} & 100 \% \quad Q' > A \\ \frac{A}{Q'} & 100 \% \quad A > Q' \end{cases}\]
2.7 RANDOM RESPONSE FUNCTION

Another approach to estimating the modal characteristics from response data is the autoregressive moving average procedure. This method has been applied to the determination of structural parameters by Gersch [44-49] and Pandit [40,41]. With this technique the response data is assumed to be caused by a white random noise input to the structure. The technique computes the best statistical model of the system in terms of its poles (from the autoregressive part), and zeros (from the moving average part), as well as statistical confidence factors on the parameters. It has been primarily used to estimate the characteristics of buildings being excited by wind forces. The data used in the computational process are the autocorrelation functions of the responses measured at various points on the structure. Since in the general case the inputs are not measured, the modal vectors are determined by referencing each response function to a single response to provide relative magnitude and phase information.
2.7.1 THEORY

The autoregressive moving average approach to the determination of structural parameters is based upon an input which is a white noise random signal within the bandwidth of interest. A second order differential equation is the basis for the developed time domain relationship.

\[(2-51) \quad [M] \ddot{\{y\}} + [C] \dot{\{y\}} + [K] \{y\} = \{f\}\]

If a structure which can be represented by Equation 2-51 is regularly sampled at each of the response functions, the resultant scalar discrete time series can be represented by the following autoregressive moving average time series model:

\[(2-52) \quad \sum_{i=0}^{2M} a(i) y(t-i) = \sum_{i=0}^{2M} b(i) x(t-i)\]

where:
- \(a(i)\) = Autoregressive parameter
- \(b(i)\) = Moving average parameter
- \(a(0) = 1\)
- \(b(0) = 1\)

This model assumes that the output \(Y\) is contaminated with an
uncorrelated zero-mean sequence of additive noise. The time series sequence set \{X\} is a zero mean uncorrelated history.

If the system transfer function is defined as follows,

\[
(2-53) \quad H(s) = \frac{F\{b(i)\}}{F\{a(i)\}} = \frac{F\{y(t)\}}{F\{x(t)\}}
\]

where: \( F\{ \} \) = Fourier transform

then Equation 2-52 is obviously just the equivalent convolution representation in the time domain. The left hand portion of Equation 2-52 is the convolution of the denominator of the transfer function with the output and the right hand portion of Equation 2-52 is the convolution of the numerator of the transfer function with the input.

The autoregressive parameters correspond to the characteristic equation of Equation 2-51. Therefore, once the autoregressive parameters are known, the damped natural frequencies and damping factors can be estimated from

\[
(2-54) \quad \sum_{i=0}^{2m} a(i) s^{2m-i} = \prod (s - e^{-p(i)\Delta t}) (s - e^{-\bar{p}(i)\Delta t})
\]

where: \( p(i) = \sigma(i) + j \omega d(i) \)

\( n = \text{Number of degrees of freedom} \)
Equation 2-54 results from the knowledge that the characteristic equation is equal to the product of all poles of the system. Poles of a realistic underdamped system always occur in complex conjugate pairs.

The zeros of Equation 2-54 are found using a complex root solving algorithm. The results will be pairs of complex zeroes \((Z, Z^*)\) from which the parameters can be found as follows:

\[
\sigma(i) \Delta t = -0.50 \log \left| \frac{Z(i)^*}{Z(i)} \right|
\]

\[
w(i) \Delta t = \arctan \left( \frac{Z(i) - Z(i)^*}{Z(i) + Z(i)^*} \right)
\]

The solution for the autoregressive moving average coefficients proceeds in a two stage least squares fashion in the Gersh solution. In the first stage a "long" auto regressive model is solved linearly by using the Yule-Walker equations. This process uses output covariance functions to determine the auto regressive coefficients based upon a determination of the order of the auto regressive model. The second stage involves setting up an equivalent moving average model for the output involving convolution of the impulse response function and the input function. This procedure also involves computations using covariance functions and results in the least squares computation...
of the moving average coefficients.

Since the solution for the autoregressive moving average coefficients and thus the structural parameter estimates are statistically based, statistical confidence factors, called coefficients of variation, for the natural frequencies and damping can be easily calculated. These coefficients represent the ratio of standard deviation of each parameter with respect to the actual parameter.

Obviously, the random response function method is complicated and variation in solution procedure exists among those using this approach. For further details or information concerning varied solution approaches, technical articles can provide further reference [42,43].

2.8 REDUCED MATRIX ESTIMATION

Over the last ten years, there has been increasing interest in being able to estimate reduced mass, stiffness, and damping matrices from experimental data. Most of these methods are based upon an indirect approach utilizing the estimated modal parameters to synthesize the reduced matrices [50, 51, 52]. This approach has not generated the anticipated results due to a
number of reasons. First of all, regardless of the approach used, the solution for the reduced matrices is not unique. There are many combinations of matrix relationships that can be generated from the given set of estimated modal parameters. Second, the reduced frequency range of the modal parameter estimates means that the matrices will be weighted to represent an incomplete model [53]. Third, the limitation of the precision of the modal parameter estimates as a result of commonly accepted experimental error tends to desensitize the process of estimating the reduced matrices. Finally, the problem of invalid modal parameter estimates will obviously result in invalid estimates of reduced matrices.

Recently, an algorithm has been developed in Germany by Link and Vollan [54] which attempts to use frequency domain input and response data to directly estimate the reduced matrices. This method has been designated Identification of Structural System Parameters (ISSPA). While this method will have some of the limitations of the indirect approach to this same problem, the published preliminary results are encouraging. Since modal parameters are found as a result of the solution of the eigenvalue problem using the reduced matrix estimations, the process of estimating the reduced matrices may represent the ultimate goal in experimental modal analysis. If this process could be correlated with a purely theoretical finite element approach, the engineering design cycle would be complete.
2.8.1 THEORY

The reduced matrix estimation method of experimental modal analysis is based primarily upon a structure undergoing testing on a shake table. The formulation of the method begins with the equations of motion for a system subjected to base excitation and force excitation.

\[
[M] \ddot{\{x\}} + [C] \dot{\{x\}} + [K] \{x\} = -[M] [H] \ddot{\{u\}} + \{f'\}
\]

where:

\{x\} = Displacement vector relative to the base displacement

\{u\} = Base displacement

\{f'\} = Force excitation

[M] = Mass matrix

[C] = Viscous damping matrix

[K] = Stiffness matrix

[H] = Transformation matrix relating base displacement to the rigid body displacement of the remaining degrees of freedom

The system is excited with harmonic base and force excitation. The steady state response is assumed to be harmonic with the same frequency. This gives the following relation:
\begin{align*}
(2-58) \quad [ -w^2[M] + jw[C] + [K] ] \{x\} &= w^2 \begin{bmatrix} [M] & [H] \{u\} + \{f\} \end{bmatrix} \\

\text{where:} \quad \{f\} &= w \{f'\}
\end{align*}

The base excitation and force excitation is chosen for frequencies where both vector quantities will have only a real part in the frequency domain. The steady-state response of \(n\) measurement points is then measured for \(m\) frequencies and stored as a complex valued frequency domain vector. If the excitation conditions are met, Equation 2-58 can be reformulated as follows:

\begin{align*}
(2-59) \quad [ -w^2(I) + jw(i) [C'] + [K'] ] \{x(i)\} &= w(i)^2 \begin{bmatrix} [H] \{u(i)\} + [M] \{f(i)\} \end{bmatrix} \\

\text{where:} \quad & [I] = \text{Identity matrix} \\
& [K'] = \text{Dynamic matrix} \\
& [K'] = [M]^{-1} [K] \\
& [C'] = \text{Transformed damping matrix} \\
& [C'] = [M]^{-1} [C]
\end{align*}

For base excitation alone, the dynamic matrix and the transformed damping matrix will be the only unknowns. If force excitation is present, the mass matrix must be known or estimated in order to proceed. If modal parameters are the desired result, the mass matrix can be chosen arbitrarily, if necessary, since the modal
parameters will depend only on the relative relationship between the matrices. In this case, the absolute characteristics of each matrix will have no meaning.

Using an estimated mass matrix, if necessary, the solution can now proceed. Equation 2-59 can be reformulated into the real and the imaginary parts.

\[
\begin{align*}
(2-60) \quad & \left[ [K'] - w(i)^2 [I] \right] \{RE(x)\} - w(i) [C'] \{IM(x)\} = w(i)^2 \left[ [H] \{u(i)\} + [M]^{-1} \{f(i)\} \right] \\
(2-61) \quad & w(i)[C'] \{RE(x)\} + \left[ [K'] - w(i)^2 [I] \right] \{IM(x)\} = \{0\}
\end{align*}
\]

The real part of the response vector is now assembled into a column of vector [A]. The imaginary part of the response vector is assembled into the corresponding column of vector [B]. The excitation vector, containing only real-valued quantities, can also be assembled into a corresponding column of vector [V]. Finally, all of the \(m\) excitation frequencies are assembled in the diagonal matrix [\(\phi\)]. Matrices [A], [B], and [V] are all \(m \times n\) matrices, with one column for each measurement point.

This results in the following equations.

\[
(2-62) \quad [A] [K']^T - [\phi^2] [A] - [\phi] [B] [C']^T = [\phi^2] [V]
\]
\[
(2-63) \quad [\phi] [A][C']^T + [B][K']^T - [\phi]^2[B] = [0]
\]

If the number of excitation frequencies \( m \) is larger than the number of measurement points \( n \), the pseudo inverse of matrices \([A]\) and \([B]\) can be calculated and Equations 2-62 and 2-63 can be solved. Since this aspect of the experiment can be easily controlled, the solution depends on the ability to perform the pseudo inverse.

Link and Vollan formulate the pseudo inverse based upon a singular value decomposition procedure under the restriction that the rank of the \([A]\) and \([B]\) matrices is equal to the effective number of degrees of freedom. This effective number of degrees of freedom is dependent upon the number of theoretical system poles in the frequency range of interest, the accuracy of the measured data, and the accuracy of the computer in the pseudo inverse procedure.

The estimation of modal parameters results from the eigenvalue problem utilizing the mass, stiffness, and damping matrices found in Equations 2-62 and 2-63. Since the transformed stiffness matrix is not symmetric, this eigenvalue problem has a right and left hand solution.

A confidence or validity check of the frequencies, damping factors, and modal vectors can be performed using a back substitution procedure. The dynamic response is calculated and
compared to the original measured response. The agreement between these responses is regarded as a measure of the accuracy of the estimated modal parameters.
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3.1 OVERVIEW

The accurate measurement of frequency response functions has gained in importance with the increasing use of experimental analysis of structures. Specifically, the frequency response function measurement has found extended use in the area of structural analysis, as applied to modeling, simulation, modal analysis, and parameter estimation. The interest in increased accuracy has been a direct result of the ability to achieve great detail in the parallel area of theoretical analysis of structures. In order to take full advantage of experimental data in the evaluation of experimental procedures and verification of theoretical approaches, the errors in measurement, generally designated noise, must be reduced to acceptable levels.

With respect to the frequency response function measurement, the errors in the estimate are generally grouped into two categories: variance and bias. The variance portion of the error is due to random deviations of each sample function from the mean.
Statistically, then, if sufficient sample functions are evaluated, the estimate will closely approximate the true function with a high degree of confidence. The bias portion of the error, on the other hand, does not necessarily reduce as a result of many samples. The bias error is due to a system characteristic or measurement procedure consistently resulting in an incorrect estimate. Therefore, the expected value is not equal to the true value. Examples of this are system nonlinearities or digitization errors such as aliasing or leakage. With this type of error, knowledge of the form of the error is vital in reducing the resultant effect in the frequency response function measurement.

3.2 FREQUENCY RESPONSE FUNCTION

The definition of the frequency response function is directly related to the concept of the transfer function. The frequency response function is the complex ratio of output response to input excitation as a function of frequency for a single input, single output situation. The definition of transfer function is essentially the same with the exception that the transfer function is a function of the complex Laplace variable s. For the specific case where the real part of the Laplace variable is
restricted to zero, the transfer function becomes equal to the frequency response function. For a multiple input, multiple output situation, the measurement process becomes very complicated [9,10,11].

Structurally, frequency response functions are normally used to describe the input-output force-displacement relationships of any system. Most often, the system is assumed to be linear and time invariant although this is not necessary. In the cases where assumptions of linearity and time invariance are not valid, the measurement of frequency response functions are also dependent upon the independent variables of time and input. In this way, a conditional frequency response function is measured as a function of other independent variables in addition to frequency.

The computation of the theoretical frequency response function depends upon the transformation of data from the time to frequency domain. The Fourier transform is used for this computation just as the Laplace transform is used to transform from the time domain to the Laplace domain. Unfortunately, though, the integral Fourier transform definition requires time histories from negative infinity to positive infinity. Since this is not possible experimentally, the computation is performed digitally using a fast Fourier transform algorithm which is based upon only a limited time history. In this way the theoretical advantages of the Fourier transform can be implemented in a
digital computation scheme.

3.3 FORMULATION OF FREQUENCY RESPONSE FUNCTION

The frequency response function can be computed in at least three ways through the use of the fast Fourier transform algorithm. In each case, the frequency response function $H$ of the system in Figure 3-1 is a function of frequency and position and all noise inputs $(K, L, M, N)$ are assumed to be zero. Therefore, the system of Figure 3-1 is reduced to that of Figure 3-2. In the following sections, three different approaches to the estimation of frequency response functions will be evaluated. In all cases, only an estimate of the frequency response function can be determined. No additional notation will be used to denote that the frequency response functions are estimates.

3.3.1 RATIO OF FOURIER TRANSFORMS

The first approach to an estimation of frequency response function is to simply divide the output by the input.
\[
\begin{align*}
X &= E + L \\
W &= X + K \\
Z &= Y + N + H' M
\end{align*}
\]

E - Desired Input  
X - True Input  
Y - True Output  
W - Measured Input  
Z - Measured Output  
K - Noise  
L - Noise  
M - Extraneous Input  
N - Noise

Figure 3-1: Frequency Response Function Measurement Schematic
(3-1) \[ H(y,x) = \frac{Y}{X} \]

where: \( H(y,x) \) = Estimate of the frequency response function

\( Y = Y(f) \) = Fourier transform of \( y(t) \)

\( X = X(f) \) = Fourier transform of \( x(t) \)

This approach is acceptable although to reduce the variance of this estimate, more sample functions must be involved. Therefore:
\[ H(y,x) = \frac{\sum_{i=1}^{M} Y(i)}{\sum_{i=1}^{M} X(i)} \]

This, though, is not acceptable unless specific relationships exist between individual sample functions. This is discussed in more detail in the classification of signal averaging. In the general case, though, Equation 3-2 provides no improvement in the estimate of frequency response and no reduction in variance when compared to Equation 3-1.

### 3.3.2 FOURIER TRANSFORM OF RATIOS

The second approach to computation of frequency response is to form estimates using a summation of the ratios of Fourier transform. Therefore,

\[ H(y,x) = \sum_{i=1}^{M} \frac{Y(i)}{X(i)} \]

This, then, preserves the phase relationship in all cases but computationally can be invalid if, in individual measurements, input at certain frequencies is essentially zero.
3.3.3 LEAST SQUARES ESTIMATION

The most reasonable approach to the estimation of frequency response is by use of a least squares technique. This is one of the standard techniques for estimating parameters of noisy signals and in this case is based upon the auto and cross power spectra.

This procedure is easy to develop and complete details are available in Appendix A. The resulting estimate of frequency response is given by:

\[ H(y,x) = \frac{G(y,x)}{G(x,x)} \]  
\[ G(y,x) = \sum_{i=1}^{M} Y(i) X(i) \]  
\[ G(x,x) = \sum_{i=1}^{M} X(i) X(i) \]  
\[ G(y,y) = \sum_{i=1}^{M} Y(i) Y(i) \]
where: $X^*$ = Complex conjugate of $X$
$Y^*$ = Complex conjugate of $Y$

This form allows the cross power spectra, $G(y,x)$, and the auto
power spectra of the input $G(x,x)$, to be counted and used to
determine an accurate estimate of frequency response even in the
presence of noise. The phase information is maintained in the
$G(y,x)$ term. In addition to this form of the frequency response,
the availability of cross and auto power spectra allows the
determination of another function, called coherence, which can
express a level of confidence with respect to the frequency
response.

The development of the estimate of the coherence function is also in
Appendix A. The pertinent result is:

$$\text{(3-8)} \quad \text{COH}(y,x) = \frac{|G(y,x)|^2}{G(x,x) G(y,x)}$$

The quantity $\text{COH}(y,x)$ is called the scalar or ordinary coherence
function and is a frequency dependent, real value between zero
and one. The coherence function describes the division of output
power into coherent and incoherent parts with respect to the
input.

When the coherence is zero, the output is caused totally by
sources other than the measured input. In general, then, the
coherence can be a measure of the degree of noise contamination in a measurement. Thus, with more averaging, the estimate of coherence may contain less variance, therefore giving a better estimate of the noise energy in a measured signal. This is not the case, though, if the low coherence is due to bias errors such as nonlinearities, multiple inputs or leakage. In all of these cases, the estimated coherence function will approach, in the limit, the expected value at each frequency, dependent upon the type of noise present in the structure and measurement system.

The coherence function indicates the degree of causality in a frequency response function. If the coherence is equal to one at any specific frequency, the system is said to have perfect causality at that frequency. In other words, the measured response power is caused totally by the measured input power (or by sources which are coherent with the measured input power). A coherence value less than unity at any frequency indicates that the measured response power is greater than that due to the measured input. This is due to some extraneous noise also contributing to the output power. It should be emphasized, however, that low coherence values does not necessarily imply poor estimates of the frequency response function, but simply means that more averaging is needed for a reliable result.

Two special cases of low coherence are worth particular mention. The first situation occurs when a leakage bias error occurs in
one or both of the input and output measurements. This causes the coherence in the area of the peaks of the frequency response to be less than unity. This error can be reduced by the use of weighting functions or by cyclic averaging to be discussed later. The second situation occurs when a significant propagation delay time occurs between the input and output as may be the case with acoustic measurements. If a propagation delay of length $t$ is compared to a sample function length of $T$, a low estimate of coherence will be estimated by the following equation [7].

\[
(3-9) \quad \text{COH}(y,x) = \text{E}[	ext{COH}(y,x)] \left[ 1 - \frac{t}{T} \right]^2
\]

where: $\text{E}[	ext{COH}(y,x)] = \text{Expected value of the ordinary coherence function}$

This propagation delay causes a bias error in the frequency response and should be removed prior to computation if possible.

In order to illustrate coherence, a graphical example of a coherence function at a specific frequency is given in Figure 3-3. Figure 3-3a shows a situation with no noise. Note that the modulus of $G(y,x)$ squared is equal to the modulus of $G(y,y)$ times the modulus of $G(x,x)$. Figure 3-3b shows the situation with a small random error added. Obviously, the modulus of $G(y,x)$ squared is less than the modulus of $G(y,y)$ times the modulus of $G(x,x)$ [6].
Figure 3-3: Graphical Representation of Coherence
3.4 FREQUENCY RESPONSE ESTIMATION ERRORS

There are several factors that contribute to the quality of actual measured frequency response function estimates. Some of the most common sources of error are due to measurement mistakes. With a proper measurement approach, most of this type of error, such as overloading the input, extraneous signal pick-up via ground loops or strong electric or magnetic fields nearby, etc., can be avoided. Violation of test assumptions are often the source of another inaccuracy and can be viewed as a measurement mistake. For example, frequency response and coherence functions have been defined as parameters of a linear system. Nonlinearities will generally shift energy from one frequency to many new frequencies, in a way which may be difficult to recognize. The result will be a distortion in the estimates of the system parameters, which may not be apparent unless the excitation is changed. One way to reduce the effect of nonlinearities is to randomize these contributions by choosing a randomly different input signal for each of the n measurements. Subsequent averaging will reduce these contributions in the same manner that random noise is reduced. Another example involves control of the system input. One of the most obvious requirements is to excite the system with energy at all
frequencies for which measurements are expected. It is important to be sure that the input signal spectrum does not have "holes" where little energy exist. Otherwise, coherence will be very low, and the variance on the frequency response function will be large.

Assuming that the system is linear, the excitation is proper, and obvious measurement mistakes are avoided, some amount of noise will be present in the measurement process. Noise is a general designation describing the difference between the true value and the estimated value. A more exact designation is to view this as the total error comprised of two terms, variance and bias. Each of these classifications are merely a convenient grouping of many individual errors which cause a specific kind of inaccuracy in the function estimate. The variance portion of the error essentially is Gaussian distributed and can be reduced by any form of synchronization in the measurement or analysis process. The bias or distortion portion of the error causes the expected value of the estimated function to be different from the true value. Normally, bias errors are removed if possible but, if the form and the source of a specific bias error is known, many techniques may be used to reduce the magnitude of the specific bias error.

Assuming that some amount of noise is always present, it is important to utilize enough measurements to improve the estimates
to some acceptable level. If sufficient averaging time is not used, the resulting estimates seem to change from one measurement to the next. These changes are effects of variance and bias and cannot be distinguished from valid estimates of frequency response or coherence. Therefore a complete understanding of the cause of variance and bias error is always desirable.

Specifically, two general categories of problems exist which may cause significant error even when great care has been taken to deal with inaccurate system assumptions and obvious measurement mistakes. The first category is concerned with the limitations of using finite information. Any measurement instrument is limited in time resolution, or frequency bandwidth. However, sampling a signal at discrete times also introduces a form of amplitude error (called aliasing) that converts high frequency energy to lower frequencies. This source of error would be classified as a bias. Thus, the time resolution and frequency bandwidth parameters are generally dictated by an anti-aliasing filter in front of the sampler. The shape of this filter influences the in-band accuracy and the stop-band rejection characteristics of the instrument. Obviously, filters are not perfect, and there is no such thing as absolute rejection. Strong signals with potential aliasing are often present to some extent. Another form of amplitude error is involved in the quantization of the analog signal to a digital signal. Since only discrete amplitude levels are possible, the amplitude will
often be in error. This source of error is normally Gaussian
distributed and therefore is part of the variance portion of the
total error.

Analogous to time resolution limits, there is always a limit on
frequency resolution. This is ultimately determined by the total
effective time over which coherent data is collected. The effect
of this finite collection time is the introduction of another
type of non-linear error (called leakage), which converts energy
at each frequency into energy within a relatively narrow band
nearby. This type of error is controlled to some extent by
weighting (or windowing) the original time domain data. However,
this type of error will always cause a considerable bias in any
portion of a measurement that is sufficiently close to a strong
signal. In the situation of excitation of lightly damped
structures, this leakage error, compared to all other sources of
error, is usually the largest bias error and often will be much
greater than the variance error. This error is basically due to
a violation of an assumption of the fast Fourier transform
algorithm. This assumption is that the true signal is periodic
within the sample period used to observe the sample function. In
the cases where both input and output are totally observable
(transient input with completely observed decay output within the
sample period) or are completely periodic integer functions of
$\Delta f$ where $\Delta f = 1/T$, there will be no contribution to the bias
error due to leakage. One advantage of cyclic signal averaging
or the zoom fast Fourier analysis is that the effective sample period is much larger. This allows for more observation time for the transient case and better frequency resolution for the continuous case. This will greatly reduce the effects of leakage. Since the same functions are included in the calculation of the coherence function, the leakage bias error must also be reduced to an acceptable level if a good statistical estimate of the coherence function is to be achieved. It should be noted, though, that the leakage bias error always causes the estimate of the frequency response to be lower at resonances than the true value[8]. The errors in the estimates of frequency response and coherence are discussed in more detail in many references [3,4,5,8].

The second category of error is concerned with signal contamination which cannot be avoided in the measurement process. A typical frequency response measurement situation is formulated in Figure 3-1. The actual input to the system is \( X \) and the actual output is \( Y \). \( E \) is the desired input which has been altered in some way such as an impedance mismatch in the excitation system. This is not serious as long as the system is essentially linear. Unfortunately, though, \( X \) and \( Y \) cannot be measured directly. Instead \( W \) is measured as the input and \( Z \) is measured as the output. In the case of the input, \( W \) differs from \( X \) due to some noise \( K \) on the input. In the case of the output, \( Z \) differs from \( Y \) due to noise on the input \( N \) as well as due to
extraneous, non-measured inputs such as M. Therefore, assuming no noise on the input:

\[(3-10) \quad Z = HX + H'X + N\]

Since neither \(H'\) nor M is measured, this term can be grouped with N as noise on the output. In this situation, though, N can no longer be assumed to be Gaussian due to the bias of \(H'\).

For the system as described, the formulation of the frequency response and coherence function relationships for the case with \(L = M = 0\) can verify the value of averaging once again.

\[(3-11) \quad W = X + K\]

\[(3-12) \quad Z = Y + N\]

Without repeating a least squares development in detail, the pertinent results are as follows:

\[(3-13) \quad H(z,w) = \frac{G(y,x) + G(n,x) + G(k,y) + G(k,l)}{G(x,x) + G(k,x) + G(k,n) + G(k,k)}\]

Naturally all noise inputs are assumed to be uncorrelated with the input and with each other, therefore:

\[(3-14) \quad H(z,w) = \frac{G(y,x)}{G(x,x) + G(k,k)}\]
But the true frequency response is:

(3-15) \[ H(y,x) = \frac{G(y,x)}{G(x,x)} \]

Therefore, the estimate of frequency response is related as follows:

(3-16) \[ H(z,w) = H(y,x) \left[ 1 + \frac{G(k,k)}{G(x,x)} \right]^{-1} \]

Therefore, if the absolute value of \(G(k,k)\) divided by \(G(x,x)\) is much less than one, the estimate of frequency response will approach the expected value. For the coherence function:

(3-17) \[ \text{COH}(z,w) = \left| \frac{G(y,x) + G(n,x) + G(k,y) + G(k,l)}{(G(x,x) + G(k,k)) (G(y,y) + G(n,n))} \right|^2 \]

With the same assumptions:

(3-18) \[ \text{COH}(z,w) = \left| \frac{G(y,x)}{G(x,x) G(y,y) + G(x,x) G(n,n) + G(y,y) G(k,k) + G(k,k) G(n,n)} \right|^2 \]

Assuming that the auto power spectra of the noise inputs are small compared to \(G(y,y)\) or \(G(x,x)\), the estimate of coherence approaches the expected value.
If the frequency response of the system to the desired input is required, an impedance isolation approach, called the instrumental variable method, can be used to remove unwanted characteristics (for example, impedance mismatch or noise on input problems) from the input spectrum. In this case, two frequency response functions are estimated and the desired frequency response computed from the estimates. First of all, the frequency response between the measured output and measured input is formulated as before.

\[(3-19) \quad H(z,w) = \frac{G(z,w)}{G(w,w)}\]

Additionally, the frequency response between the desired input and measured input is computed.

\[(3-20) \quad H(e,w) = \frac{G(e,w)}{G(w,w)}\]

From these two quantities, the desired frequency response function can be estimated.

\[(3-21) \quad H(z,e) = \frac{H(z,w)}{H(e,w)} = \frac{G(z,w)}{G(e,w)}\]

Where extreme system dynamics are involved, this scheme may contain computational problems but for the case of unbiased noise
on the input or a small impedance mismatch between the input and system, little difficulty should be encountered. If the problem arises solely from an impedance mismatch, the desired input $E$ can be altered through a convolution process involving the unit impulse response between $E$ and $W$. This will compensate $E$ so that the true input $X$ contains the desired spectrum.

3.5 REDUCTION OF ERRORS

Four different approaches can be used to reduce the error involved in frequency response function measurements in current fast Fourier transform (FFT) analyzers. The use of averaging can significantly reduce errors of both variance and bias and is probably the most general technique in the reduction of errors in frequency response function measurement. Selective excitation is often used to verify nonlinearities or randomize characteristics. In this way, bias errors due to system sources can be reduced or controlled. The increase of frequency resolution through the zoom fast Fourier transform can improve the frequency response function estimate primarily by reduction of the leakage bias error due to the use of a longer time sample. The zoom fast Fourier transform by itself is a linear process and does not involve any specific error reduction characteristics compared to
a baseband fast Fourier transform (FFT). Finally, the use of weighting functions (windows) is widespread and much has been written about their value [1,2,3]. Primarily, weighting functions compensate for the bias error (leakage) caused by the analysis procedure.

3.5.1 SIGNAL AVERAGING

The averaging of signals is normally viewed as a summation or weighted summation process where each sample function has a common abscissa. Normally, the designation of "history" is given to sample functions with the abscissa of absolute time and the designation of "spectrum" is given to sample functions with the abscissa of absolute frequency. The spectra are normally generated by Fourier transforming the corresponding history. In order to generalize and consolidate the concept of signal averaging as much as possible, the case of relative time could also be considered. In this way "relative history" could be discussed with units of the appropriate event rather than seconds and a "relative spectrum" would be the corresponding Fourier transform with units of cycles per event. This concept of signal averaging is used widely in structural signature analysis where the event is a revolution. This kind of approach simplifies the application of many other concepts of signal relationships such
as Shannon's sampling theorem and Rayleigh's criterion of frequency resolution.

The process of signal averaging as it applies to frequency response functions is simplified greatly by the intrinsic uniqueness of the frequency response function. Since the frequency response function can be expressed in terms of system properties of mass, stiffness, and damping, it is reasonable to conclude that in most realistic structures, the frequency response functions are considered to be constants just like mass, stiffness, and damping. This concept means that when formulating the frequency response function using:

\[ H(y,x) = \frac{G(y,x)}{G(x,x)} \]  

(3-22)

only the estimate of frequency response is necessarily unique. \( G(y,x) \) and \( G(x,x) \) are only unique as individual quantities when the input is statistically in agreement with the procedure used to develop Equation 3-4. Therefore, the estimate of frequency response is valid whether the input is stationary, non-stationary, or deterministic. In general, the individual terms \( G(y,x) \) and \( G(x,x) \) are statistically meaningful only if the input is stationary.

The concept of the intrinsic uniqueness of the frequency response function also permits a greater freedom in the testing procedure.
Each sample function can be derived as a result of a separate test or as the result of different portions of the same continuous test situation. In either case, the estimate of frequency response function will be the same.

3.5.1.1 SIGNAL AVERAGING TECHNIQUES

The approaches to signal averaging vary only in the relationship between each sample function used. Since the Fourier transform is a linear function, there is no theoretical difference between the use of histories or spectra. (Practically, though, there are precision considerations which will be discussed later). With this in mind, the signal averaging useful to frequency response function measurements can be divided into three classifications:

1) Asynchronous
2) Synchronous
3) Cyclic

These three classifications refer to the trigger and sampling relationships between sample functions.

The classification of asynchronous signal averaging refers to the case where no known relationship exists between individual sample functions except for the intrinsic uniqueness of the frequency response function. In this case the least squares approach to the estimate of frequency response must be used since no other
way of preserving phase and improving the estimate is available. In this situation, the trigger for digitization (sampling and quantization) takes place in a random fashion dependent only upon the equipment availability. The digitization is said to be in a free-run mode.

The synchronous classification of signal averaging adds an additional constraint that each sample function must be initiated with respect to a system input. This fact, together with the intrinsic uniqueness, would allow the frequency response function to be formed as a summation of ratios of $Y$ divided by $X$ since phase is preserved. Even so, the reduction of variance and the value of coherence available with the power spectra approach would preclude the use of any other technique in most cases. The ability to synchronize the initiation of digitization allows for use of non-stationary or deterministic inputs with a resulting increased signal to noise ratio and reduced leakage. Both of these improvements in the frequency response function estimate are due to more of the input and output being observable in the limited time window.

The synchronization takes place as a function of a trigger signal occurring in the input (internally) or in some event related to the input (externally). An example of an internal trigger would be the case where an impulsive input is used to estimate the frequency response. All sample functions would be initiated when
the input reached a certain amplitude and slope. A similar example of an external trigger would be the case where the impulsive excitation to a speaker is used to trigger the estimate of frequency response between two microphones in the sound field. Again, all sample functions would be initiated when the trigger signal reached a certain amplitude and scope.

The cyclic classification of signal averaging involves the added constraint that the digitization is coherent between sample functions. This means that the exact time (absolute or relative) between each sample function is used to enhance the signal averaging process. Rather than trying to keep track of elapsed time between sample functions, the normal procedure is to allow no time to elapse between successive sample functions. This process can be described as a comb digital filter in the frequency domain with the teeth of the comb at frequency increments dependent upon the periodic nature of the sampling with respect to the event measured. The result is an attenuation of the spectrum between the teeth not possible with other forms of averaging. Application of this technique is detailed in Appendix B.

3.5.1.2 SIGNAL AVERAGING PRECISION

Precision considerations with respect to signal averaging are mostly concerned with the word size of the analog to digital
converter (ADC) and the word size of the mini-computer. Problems may occur when many averages are used to estimate either the frequency response or coherence function. For most structural test situations, signal averaging involving less than two hundred sample functions will rarely be a source of significant error if proper digitization procedures are considered.

Three considerations are very important with respect to efficient digitization and signal averaging. All three depend upon the concept of actual word size versus effective word size. The actual word size is the number of bits available in each word in the ADC or computer. The effective word size is the number of bits used in each word in any operation in the ADC or computer. The first consideration is that of dynamic range. This has to do with the input to the ADC in the digitization process. Since the actual word size of the ADC is fixed, the dynamic range of the ADC (60 db for 10 bits, 72 db for 12 bits) is only meaningful if the quantization of an input signal of interest involves all of the bits of the actual word. Two situations may exist where this is not true. First, if the input ranges for the ADC are not automatically or manually set to the optimum position, some loss of dynamic range will occur. This means that the maximum level of the data should not be less than one half of the input voltage range. Secondly, if the signal has more dominant information content outside the band of interest, a significant portion of the dynamic range will be used to describe the unwanted
characteristic. This will reduce the potential dynamic range available to the portion of the signal in which there is interest. Some common examples of this are a large mean value offset or large harmonic component (such as 60 Hertz) as shown in Figure 3-4. Both of these situations cause the effective word size of the ADC to be smaller than the actual word size of the ADC with respect to the information of interest.

The second consideration is that of processor gain. If the actual word size of the computer is greater than the actual word size of the ADC, signal averaging will produce finer resolution than one part in 2 raised to the power n where n is the actual word size of the ADC. This process is demonstrated in Figure 3-5 for the case of an actual ADC word size of three bits, an actual computer word size of five bits, and two averages. The factor of 0.25 is the scaling (due to the factor of 2 raised to the power m where m is the difference in actual word sizes) necessary when the ADC word is shifted to the most significant three bits of the computer word. In this case, the effective word size of the ADC is greater than the actual word size.

The third consideration involves the concept of alignment between an individual sample function and the accumulated summation. Whether the signal averaging proceeds in a linear fashion or a stable fashion, at some point the accumulated summation will be large compared to the individual sample function into the
Figure 3-4: Effective Word Size Less Than Actual Word Size
THREE BIT ADC

Output Code

Input Voltage

<table>
<thead>
<tr>
<th>ADC</th>
<th>COMPUTER</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3 = 011</td>
<td>01100 = 12</td>
<td>0.25</td>
</tr>
<tr>
<td>2 = 010</td>
<td>01000 = 8</td>
<td>0.25</td>
</tr>
<tr>
<td>5 = 101</td>
<td>10100 = 20</td>
<td>0.25</td>
</tr>
</tbody>
</table>

sum

2 = 010 01010 = 10 0.25 avg.

\[ \text{2.0} \quad \text{2.5} \]

Figure 3-5: Effective Word Size Greater Than Actual Word Size
accumulated summation, the sample must be first scaled the same as the accumulated summation. As this occurs, since only a limited number of bits are available in the computer word, bits representing significant information are shifted to the right and lost. This can also be termed "underflow". In this case, the effective word size of each individual sample is smaller than the actual word size once this sort of scaling occurs. One way to avoid this problem is to always average summations with like precision. In the case of averaging time history information, the difference in actual ADC word size and actual computer word size may be six bits (ten bits compared to sixteen bits). This means that up to sixty-four (2 raised to the power 6) histories may be averaged before scaling occurs. Therefore, if more histories are required, sets of sixty-four histories should first be formed and an average obtained for each set. In order to average these sets of partial averages, or any quantities involving the same number of bits and like precision, each time two terms are added, scaling may occur. Therefore, the final summation in this case should proceed on a two by two basis until only one term is left. In this way the maximum possible precision is maintained. This technique for maintaining precision is called multistage processing.
3.5.1.3 OVERLAPPING HISTORIES

There are at least two common averaging techniques that use histories which may or may not overlap. In both cases, the averaging techniques involve processing random data histories in order to enhance the data. The first case is that of overlap processing. Overlap processing involves using individual sample histories which are not totally independent from one another. The dependence that occurs results from each successive history starting before the previous history ends. This process is described by the following equation.

\[(3-23) \quad x(T) = \frac{1}{N} \sum_{i=1}^{N} y(t(i) + T)\]

where: \( x(T) = \) Overlap processed history
\( y(t) = \) Random data history
\( N = \) Number of averages
\( t(i) = (i-1) \cdot OVF \cdot T \)
\( T = \) Sample period
\( OVF = \) Overlap factor

The overlap factor used in Equation 3-23 is computed from the following relationship.

\[(3-24) \quad OVF = \frac{T - t(1)}{T}\]
It should be obvious that in the general case, this process does not involve any new data and, therefore, statistically does not improve the estimation process. In the special case where weighting functions are involved, this technique can utilize data that is otherwise ignored. Figure 3-6 is an example of a data history that has been weighted to reduce the leakage error. The data prior to twenty percent of each sample period and after eighty percent of each sample period is nearly eliminated by the Hanning window used. Using an overlap factor of at least twenty to thirty percent as in Figure 3-7 involves this data once again in the averaging process.

The second case involving overlapping histories is that of random decrement analysis [14-17]. This process involves the overlapping of histories in order to enhance the deterministic portion of the random record. In general, the random response data can be considered to be made up of two parts: a deterministic part and a random part. Averaging in the time domain, the random part can be reduced if a trigger signal with respect to the information of interest exists. In the previous discussions, this trigger signal has been a function of the input (asynchronous or synchronous averaging) or of the sampling frequency (cyclic averaging). More generally, though, the trigger function can be any function with characteristics related to the response history. Specifically, then, the random decrement technique utilizes the assumption that the
deterministic part of the random response signal itself contains free decay step and impulse response functions and can be used as the trigger function. Therefore, by starting each history at a specific value and slope of the random response function, characteristics related to the deterministic portion of the history will be enhanced. The following equation describes the averaging process.

\[
(3-25) \quad x(\tau) = \frac{1}{N} \sum_{i=1}^{N} y(t(i) + \tau)
\]

where: \( y(t) = \) Random response history  
\( x(\tau) = \) Random decrement history  
\( N = \) Number of averages

There are three specific cases of Equation 3-19 that represent the limiting results of its use [18]. The three cases involve ways to choose the starting values, \( t(i) \). The first case occurs when each starting value is chosen when the random response history reaches a specific constant level with alternating slopes for each successive starting value. The random decrement history for this case becomes the free decay step response function. An example of this case for the first few averages is shown in Figure 3-8.

The second case occurs when each starting value is chosen when the random response history crosses the zero axis with positive
Figure 3-8: Random Decrement Signal Processing
slope. The random decrement history for this case becomes the free decay positive impulse response function.

The third case occurs when each starting value is chosen when the random response history crosses zero with negative slope. The random decrement history for this case becomes the free decay negative impulse response function.

Therefore, in each of these cases, the random decrement technique acts like a notched digital filter with pass bands at the poles of the trigger function. This tends to eliminate spectral components not coherent with the trigger function.

If a secondary function is utilized as the trigger function, only the history related to the poles of the secondary function will be enhanced by this technique. If the trigger function is sinusoidal, the random decrement history will contain information related only to that sinusoid. Likewise, if the trigger function is white noise, the random decrement history will be a unit impulse function at time zero. One useful example of this concept was investigated for conditioning random response histories so that information unrelated to the theoretical input history is removed. In this situation, the theoretical input history serves as the trigger function. The random decrement history formed on the basis of this trigger function represents the random response function that would be formed if the theoretical input history were truly the system input. In
reality, the measured input history may vary due to noise, impedance mismatch, etc.

3.5.2 SELECTIVE EXCITATION

Inputs which can be used to excite a system in order to determine frequency response functions belong to one of two classifications. The first classification is that of a random signal. Signals of this form can only be defined by their statistical properties over some time period. Any subset of the total time period is unique and no explicit mathematical relationship can be formulated to describe the signal. Random signals can be further classified as stationary or non-stationary. Stationary random signals are a special case where the statistical properties of the random signals do not vary with respect to translations with time. Finally, stationary random signals can be classified as ergodic or non-ergodic. A stationary random signal is ergodic when a time average on any particular subset of the signal is the same for any arbitrary subset of the random signal. All random signals which are commonly used as input signals fall into the category of ergodic, stationary random signals.

The second classification of inputs which can be used to excite a system in order to determine frequency response functions is that
of a deterministic signal. Signals of this form can be represented in an explicit mathematical relationship. Deterministic signals are further divided into periodic and non-periodic classifications. The most common inputs in the periodic deterministic signal designation are sinusoidal in nature while the most common inputs in the non-periodic deterministic designation are transient in form.

The choice of input to be used to excite a system in order to determine frequency response functions depends upon the characteristics of the system, upon the characteristics of the parameter estimation, and upon the expected utilization of the data. The characterization of the system is primarily concerned with the linearity of the system. As long as the system is linear, all input forms should give the same expected value. Naturally, though, all real systems have some degree of nonlinearity. Deterministic input signals result in frequency response functions that are dependent upon the signal level and type. A set of frequency response functions for different signal levels can be used to document the nonlinear characteristics of the system. Random input signals, in the presence of nonlinearities, result in a frequency response function that represents the best linear representation of the nonlinear characteristics for a given level of random signal input. For small nonlinearities, use of a random input will not differ greatly from the use of a deterministic input.
The characterization of the parameter estimation is primarily concerned with the type of mathematical model being used to represent the frequency response function. Generally, the model is a linear summation based upon the modal parameters of the system. Unless the mathematical representation of all nonlinearities is known, the parameter estimation process cannot properly weight the frequency response function data to include nonlinear effects. For this reason, random input signals are prevalently used to obtain the best linear estimate of the frequency response function when a parameter estimation process using a linear model is to be utilized.

The expected utilization of the data is concerned with the degree of detailed information required by any post-processing task. For experimental modal analysis, this can range from implicit modal vectors needed for trouble-shooting to explicit modal vectors used in an orthogonality check. As more detail is required, input signals, both random and deterministic, will need to match the system characteristics and parameter estimation characteristics more closely. In all possible uses of frequency response function data, the conflicting requirements of the need for accuracy, equipment availability, testing time, and testing cost will normally reduce the possible choices of input signal.

With respect to the reduction of the variance and bias errors of the frequency response function, random or deterministic signals
can be utilized most effectively if the signals are periodic with respect to the sample period or totally observable with respect to the sample period. If either of these criteria are satisfied, regardless of signal type, the predominant bias error, leakage, will be eliminated. If these criteria are not satisfied, the leakage error may become significant. In either case, the variance error will be a function of the signal-to-noise ratio and the amount of averaging.

Many signals are appropriate for use in experimental modal analysis. Some of the most commonly used signals are described in the following sections [19,22].

3.5.2.1 SLOW SWEPT SINE

The slow swept sine signal is a periodic deterministic signal with a frequency that is an integer multiple of the FFT frequency increment. Sufficient time is allowed in the measurement procedure for any transient response to the changes in frequency to decay so that the resultant input and response histories will be periodic with respect to the sample period. Therefore, the total time needed to compute an entire frequency response function will be a function of the number of frequency increments required and the system damping.

There are many important characteristics of the slow swept sine
signal relevant to its use with respect to experimental modal analysis. Historically, this signal was the only possible choice compatible with the available hardware. This signal has the highest signal to noise ratio used signal. Since both the input and output histories are periodic with respect to the sample period, the leakage bias error is eliminated. This signal provides the best basis for documenting nonlinear characteristics of a system but the form of the resultant frequency response function will not be compatible with most modal parameter estimation algorithms which are based upon a linear model. Finally, for a fixed number of simultaneous inputs, the time required per measurement for this signal will normally be a factor of ten or twenty times slower than any other signal type.

3.5.2.2 PERIODIC CHIRP

The periodic chirp is a fast swept sine signal that is a periodic deterministic signal and is formulated by sweeping a sine signal up or down within a frequency band of interest during a single sample period. Normally, the fast swept sine signal is made up of only integer multiples of the FFT frequency increment. This signal is repeated without change so that the input and output histories will be periodic with respect to the sample period. There are many important characteristics of the fast swept sine signal relevant to its use with experimental modal analysis.
Obviously, the peak to RMS energy ratio and signal to noise ratio compare favorably to the slow swept sine signal. The amount of time involved for a fixed number of simultaneous inputs is minimized while still producing a leakage free measurement. The energy input at any given frequency is lower than with the slow swept sine but noise not coherent with the input can be averaged out. Noise coherent with the input, typically as a result of the rattle produced by loose components, or nonlinearities will be smoothed in the resultant measurement due to the smooth phase angle of the fast swept sine signal. This is an advantage over other signals with random phase with respect to the modal parameter estimation.

3.5.2.3 IMPACT

The impact signal is a transient deterministic signal which is formed by applying an input pulse to a system lasting only a very small part of the sample period. The width, height, and shape of this pulse will determine the usable spectrum of the impact. Briefly, the width of the pulse will determine the frequency spectrum while the height and shape of the pulse will control the level of the spectrum. Impact signals have proven to be quite popular due to the freedom of applying the input with some form of an instrumented hammer. While the concept is straightforward, the effective utilization of an impact signal is very involved.
[20, 21].

There are many important characteristics of the impact signal relevant to its use with experimental modal analysis. Obviously the setup, equipment, and fixturing required for this signal type is the least of any signal. Experience has shown that for complete modal surveys, though, one of the other signal types will give comparable results in equal or less time. Often, though, this signal type is usable when no other signal type is possible or practical. The peak to RMS energy ratio is quite high but is not repeatable between signals in the common application of this signal type. This signal type is, therefore, not useful for nonlinear systems. The signal to noise ratio of the impact signal is low and, thus, the impact signal is most useful in low noise environments. Although the input is totally observed within the sample period, the output may or may not be totally observed depending upon the system damping. If the output is not totally observed, the leakage error will become significant. Finally, the problems with leakage, certain sources of noise, and overloading of the input signal can be successfully reduced but this will require greater attention and more experience than with other signal types.

3.5.2.4 STEP RELAXATION

The step relaxation signal is a transient deterministic signal
which is formed by releasing a previously applied static input. The sample period begins at the instant that the release occurs. This signal is normally generated by the application of a static force through a cable. The cable is then cut or allowed to release through a shear pin arrangement.

There are many important characteristics of the step relaxation signal relevant to its use with experimental modal analysis. The frequency spectrum of this type of signal is not linear and weights the low frequency range heavily. For this reason, it is particularly applicable to large systems that cannot be easily tested with any other signal type. The setup required of this signal type, though, is greater than any other signal type. While the input history is totally observable within the sample period, the output history has the same characteristics as the impact signal with the same possible leakage errors[23].

3.5.2.5 PURE RANDOM

The pure random signal is an ergodic, stationary random signal which has a Gaussian probability distribution. In general, the frequency content of the signal contains all frequencies (not just integer multiples of the FFT frequency increment) but may be filtered to include only information in a frequency band of interest. The measured input spectrum of the pure random signal will be altered by any impedance mismatch between the system and
the exciter.

There are many important characteristics of the pure random signal relevant to its use with experimental modal analysis. Since neither the input nor response signal is periodic with respect to the sample period, the leakage error will be present. Normally, weighting functions are used to try to reduce this error. This signal type results in the best linear approximation of any system nonlinearities for a controlled level of pure random signal. The amount of time required per measurement for a fixed number of simultaneous inputs will be minimal and is probably the primary asset of this signal type. The peak to RMS energy ratio is sufficiently low so that most problems with noncoherent noise can be eliminated through averaging.

3.5.2.6 PSEUDORANDOM

The pseudorandom signal is an ergodic, stationary random signal consisting only of integer multiples of the FFT frequency increment. The frequency spectrum of this signal has a constant amplitude with random phase. If sufficient time is allowed in the measurement procedure for any transient response to the initiation of the signal to decay, the resultant input and response histories are periodic with respect to the sample period. The number of averages used in the measurement procedure is only a function of the reduction of the variance error. In a
noise free environment, only one average may be necessary.

There are many important characteristics of the pseudorandom signal relevant to its use with experimental modal analysis. The time required per measurement is minimized as well as eliminating the leakage error. The peak to RMS energy ratio is sufficiently low so that most problems with noncoherent noise can be eliminated through averaging. Unfortunately, though, if the system contains loose components, these components will produce a rattle coherent with the repetitive input. This type of noise cannot be averaged out and will contribute to later difficulties during the modal parameter estimation process.

3.5.2.7 PERIODIC RANDOM

The periodic random signal is an ergodic, stationary random signal consisting only of integer multiples of the FFT frequency increment. The frequency spectrum of this signal has random amplitude and random phase distribution. Since a single history will not contain information at all frequencies, a number of histories must be involved in the measurement process. For each average, an input history is created with random amplitude and random phase. The system is excited with this input in a repetitive cycle until the transient response to the change in excitation signal decays. The input and response histories should then be periodic with respect to the sample period and are
recorded as one average in the total process. With each new average, a new history, uncorrelated with previous input signals, is generated so that the resulting measurement will be completely randomized.

There are many important characteristics of the periodic random signal with respect to its use with experimental modal analysis. The periodic random signal determines the best estimate of a linear frequency response function consistent with current modal parameter estimation algorithms. Although the time required per measurement is longer than with other signal types (except for slow swept sine), the elimination of the leakage error is a good return for the time invested. When the test duration is not the most critical concern, a periodic random signal will give the best possible result.

3.5.2.8 RANDOM TRANSIENT

The random transient signal is neither a completely transient deterministic signal nor a completely ergodic, stationary random signal but contains properties of both signal types. The frequency spectrum of this signal has random amplitude and random phase distribution and contains energy throughout the frequency spectrum. The difference between this signal and the periodic random signal is that the random transient history is truncated to zero after some percentage of the sample period (normally
fifty to eighty percent). The measurement procedure duplicates the periodic random procedure but without the need to wait for the transient response to decay. The point that the input history is truncated is chosen so that the response history decays to zero within the sample period. Even for lightly damped systems, the response history will decay to zero very quickly due to the damping provided by the exciter system trying to maintain the input at zero. This damping, provided by the exciter system, is often overlooked in the analysis of the characteristics of this signal type. Since this measured input, although not part of the generated signal, includes the variation of the input during the decay of the response history, the input and response histories are totally observable within the sample period and the system damping is unaffected.

There are many important characteristics of the random transient signal relevant to its use with experimental modal analysis. The signal to noise ratio is much larger as well as the peak to RMS energy ratio being much lower than other forms of transient signals. The random aspect of the signal type does provide the best linear representation of the frequency response function consistent with the requirements of modal parameter estimation algorithms. In general, the random transient signal has the same characteristics as the periodic random signal with a considerable reduction in the time required per measurement. Additionally, though, since the frequency spectrum is continuous, the zoom fast
Fourier transform can be used to increase the frequency resolution. When the test duration is one of the critical test concerns, a random transient signal will give the best possible result.

3.5.3 INCREASED FREQUENCY RESOLUTION

An increase in the frequency resolution of a frequency response function effects measurement errors in several ways. Obviously, finer frequency resolution allows more exact determination of the damped natural frequency of each modal vector. The increased frequency resolution means that the level of a broadband signal is reduced. The most important benefit of increased frequency resolution, though, is a reduction of the leakage error. Since the distortion of the frequency response function due to leakage is a function of frequency spacing, not frequency, the increase in frequency resolution will reduce the true bandwidth of the leakage error centered at each damped natural frequency. In order to increase the frequency resolution, the total time per history must be increased in direct proportion. The longer data acquisition time will increase the variance error problem when transient signals are utilized for input as well as emphasizing any nonstationary problem with the data. The increase of frequency resolution will often require multiple acquisition
and/or processing of the histories in order to obtain an equivalent frequency range. This will increase the data storage and documentation overhead as well as extending the total test time.

There are two approaches for increasing the frequency resolution of a frequency response function. The first approach involves increasing the number of spectral lines in a baseband measurement. The advantage of this approach, is that no additional hardware or software is required. Often, FFT analyzers do not have the capability to alter the number of spectral lines used in the measurement. The second approach involves the reduction of the bandwidth of the measurement while holding the number of spectral lines constant. If the lower frequency limit of the bandwidth is always zero, no additional hardware or software is required. Ideally, though, for an arbitrary bandwidth, hardware and/or software to perform a zoom FFT will be required.

The zoom FFT process for computing the frequency response function has additional characteristics pertinent to the reduction of errors. Primarily, more accurate information can be obtained on weak spectral components if the bandwidth is chosen to avoid strong spectral components. The out-of-band rejection of the zoom FFT is better than most analog filters that could be used in a measurement procedure to attempt to achieve the same
results. Additionally, the precision of the resulting frequency response function will be improved due to processor gain inherent in the zoom FFT calculation procedure. An example of the improvement of the frequency response function using a zoom FFT can be seen in Figure 3-9.

3.5.4 WEIGHTING FUNCTIONS

Weighting functions, or data windows, are probably the most common approach to the reduction of the leakage error in the frequency response function. While weighting functions are sometimes desirable and necessary, weighting functions are often utilized when one of the other approaches to error reduction would give superior results. Averaging, selective excitation, and increasing the frequency resolution all act to reduce the leakage error by the elimination of the cause of the error. Weighting functions, on the other hand, attempt to compensate for the leakage error after the fact. This compensation for the leakage error causes an attendant distortion of the frequency and phase information of the frequency response function, particularly in the case of closely spaced, lightly damped system poles. This distortion is a direct function of the width of the main lobe and the size of the side lobes of the spectrum of the weighting function. Examples of some common weighting functions
Figure 3-9: Increased Frequency Resolution
are given in Figure 3–10. Complete details concerning these and many other weighting functions are available from many sources [1,2,3,8].

Weighting functions may be applied to all classifications of signal averaging. The most common case is a weighting function equal to the inverse of the number of averages used in the estimate. When this weight is used, the individual power spectra can be weighted at the end of the signal averaging or as an ongoing procedure referred to as stable averaging. This type of weighting introduces no further distortion in the frequency response function estimate but, also, does not act to compensate for the leakage error.
Figure 3-10: Typical Weighting Functions
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4.1 OVERVIEW

The estimation of frequency and damping as part of the modal parameter process is dependent upon many criteria. Obviously, the frequency response function information must be free of all variance and bias errors with respect to the frequency increment used to measure the data. At this point, though, a degree of variability is involved as to the algorithm and the model that is to be used as a basis for the calculation of the complex system poles. The algorithm used to implement the model may involve a linear closed form solution for the frequency and damping independent of the modal vectors or may involve the frequency, damping, and modal vector information in a nonlinear iterative solution process. The model that is to be used in either algorithm may involve single degree of freedom concepts or multiple degree of freedom concepts with or without the benefit of residuals to account for characteristics outside the measured frequency range of interest. If the algorithm is based upon a
nonlinear iterative solution process, the choice of a model that does or does not include the concept of complex modal vectors will effect the resultant estimate of frequency and damping. Finally, the ability to determine the number of degrees of freedom in the data becomes the overriding consideration in all algorithms involving more than one degree of freedom. The evaluation of these as well as other criteria is dependent upon the expertise of the operator involved with this phase of the experimental modal analysis.

In order to improve the estimates of frequency and damping, two potential directions can be taken. First of all, better modal parameter estimation algorithms can be developed. This is in itself a separate dissertation topic as demonstrated by current work by Mergeay at the Catholic University of Leuven, Belgium. Based upon personal experience, though, current modal parameter estimation algorithms will generally provide accurate estimates if used within their limitations. Therefore, it is likely that only marginal improvement can be made by further work in this area at the present time.

The second direction that can be followed is to develop techniques which can complement current modal parameter estimation algorithms. These techniques can be oriented toward providing new methodology and correlation criteria which can be used in conjunction with available experimental modal analysis
methods, specifically the modal parameter estimation phase of the frequency response function method. With the rapid expansion of the area of experimental modal analysis, it is apparent that techniques which permit the use of current modal parameter estimation algorithms but reduce the required expertise of the operator would be extremely beneficial.

The concept of the enhanced frequency response function can be used to provide a method for estimating frequency and damping values with more accuracy. The enhanced frequency response function is a weighted average of all or part of the frequency response function matrix. The weighting function is some estimate of the modal vector associated with the frequency and damping (system pole). An eigenvector, resulting from the theoretical model of the system, can be used as the estimate of the modal vector. When this approach is used, the enhanced frequency response function can be used to verify the finite element model.

4.2 ENHANCED FREQUENCY RESPONSE FUNCTION

The theoretical concept of the forced normal mode excitation method of experimental modal analysis can be used to manipulate frequency response functions so as to enhance a particular mode.
of vibration mathematically. An enhanced frequency response function is created by computing a linear combination of a set of frequency response functions that have either a common response location or a common exciter location. The coefficients of this linear combination correspond to a forcing vector applied to the structure at the varying locations associated with the set of frequency response functions used. If the variation is in the exciter location, this correspondence is direct; if the variation is in the response locations, then the correspondence depends on the validity of assuming that the structure has a symmetric frequency response function matrix (that Maxwell-Betti reciprocity is true for the structure).

When the weighting vector, composed of the coefficients of the linear combination of a set of frequency response functions, is approximately equal to the components of the modal vector for the mode to be enhanced, the mode of interest is weighted more strongly than the other modes, and the enhanced frequency response function appears much more like the frequency response function of a structure having a single degree of freedom. This enhancement technique requires that the frequency response functions in the linear combination be consistent. This means that the frequency response functions must have the same theoretical system pole for the mode to be enhanced, and that the weighting vector correspond closely to the modal vector. If this is not the case, the enhanced frequency response function will
include distortions that prevent the modal parameter estimation algorithms from computing a proper estimate.

4.2.1 THEORY

The frequency response of a mechanical system can be expressed in terms of its modal parameters as follows:

\[
H(i,j,w) = \sum_{r=1}^{\infty} \frac{a(i,j,r)}{s - p(r)} + \frac{\alpha(i,j,r)}{s - p^*(r)}
\]

where:
- \( i \) = Response Point
- \( j \) = Excitation Point
- \( r \) = Mode Number
- \( p(r) \) = Complex Pole
- \( p(r) = \sigma(r) + wd(r) \)
- \( wd(r) \) = Damped Natural Frequency for mode \( r \)
- \( \sigma(r) \) = Damping Factor for mode \( r \)
- \( a(i,j,r) \) = Complex Residue for mode \( r \)

Equation 4-1 can be reformulated into an equivalent modal parameter estimation model under the added restriction of real modes [2]. This model can be stated as follows:
(4-2) \[ H(i,j,w) = \sum_{r=1}^{m} \frac{U(i,r) U(j,r)}{M(r) \left[ w^2 - wn(r)^2 + 2j \sigma(r) w \right]} \]

where: wn(r) = Undamped natural frequency of mode r
M(r) = Generalized mass of mode r
U(i,r) = Modal coefficient for mode r at point i

At this point, part of Equation 4-2 can be defined for convenience as:

(4-3) \[ K(r,w) = \frac{U(j,r)}{M(r) \left[ w^2 - wn(r)^2 + 2j \sigma(r) w \right]} \]

Equation 4-3 is a function of excitation position j. Since, within a column of the frequency response function matrix, this term is a constant, the notation for enhanced frequency response function should include some designation indicating this, such as K(r,w,j). This additional notation will be dropped for the sake of simplicity. Then, Equation 4-2 can be restated as follows:

(4-4) \[ H(i,j,w) = \sum_{r=1}^{m} K(r,w) U(i,r) \]

Assuming that the modal vectors are known, then from a set of experimental data, the measured frequency response function must be equal to linear superposition of these modal vectors of the system as indicated in Equation 4-4. It should be possible to
determine the functions $K(r,w)$ by using a pseudo-inverse procedure. This can be done at each frequency using all the measured data (for all values of $i$) and the assumed modal vectors. If the measured frequency responses are actually composed of the assumed modal vectors, then the function $K(r,w)$ should be equivalent to the frequency response of the single degree of freedom system as indicated by Equation 4-3. The complex pole of the single degree of freedom equation is the complex pole of the mode $r$. Therefore, the function $K(r,w)$ can be used to determine the complex pole of mode $r$. The curve $K(r,w)$ is referred to as the enhanced frequency response function for mode $r$. The enhanced frequency response function will contain information related to the modal vector that is used in the enhancement process as well as information related to any modal vector that is linearly related to that modal vector. If the assumed modal vectors are not the true modal vectors of the system, then the enhanced frequency response function will appear to have several degrees of freedom. The resulting function will still be useful in order to obtain a better estimate of the complex pole than will be found using any of the measured frequency response functions.

A pseudo-inverse procedure can be used to estimate the enhanced frequency response function for mode $r$. One possible technique is to compute the weighted least squares estimate of this enhanced frequency response function.
The weighted error term for the i-th point:

\[ E(i,w) = W(i) \left[ H(i,j,w) - \sum_{r=1}^{m} K(r,w) U(i,r) \right] \]

The total weighted error is:

\[ E = \sum_{i=1}^{m} W(i)^2 \left| E(i,w) \right|^2 \]

\[ E = \sum_{i=1}^{m} W(i)^2 \left[ H(i,j,w) - \sum_{r=1}^{m} K(r,w) U(i,r) \right] \]

Taking derivatives with respect to the complex conjugate of the enhanced frequency response function, the k-th normal equation becomes:

\[ \frac{\partial E}{\partial K^{*}(k,w)} = 0 \]

\[ 0 = \sum_{i=1}^{m} W(i)^2 \left[ H(i,j,w) - \sum_{r=1}^{m} K(r,w) U(i,r) \right] \]

\[ \left[ -U^{*}(i,k) \right] \]
or

\[
\sum_{\zeta=1}^{m} W(i)^2 H(i,j,w)^* U(i,k) = \\
\sum_{\zeta=1}^{m} \sum_{r=1}^{m} W(i)^2 K(r,w) U(i,r)^* U(i,k)
\]

In matrix form:

\[
\{A\} = [B] \{C\}
\]

where:

\[
A(k) = \sum_{\zeta=1}^{m} W(i)^2 H(i,j,w)^* U(i,k)
\]

\[
B(k,r) = \sum_{\zeta=1}^{m} W(i)^2 U(i,r)^* U(i,k)
\]

\[
C(r) = K(r,w)
\]

The term \(B(k,r)\) is the weighted inner product of the modal vectors. If the weighting function is chosen properly and the presumed modal vectors form a linear set, then the inner product has the property of an orthogonal function.

\[
(4-10a) \quad \sum_{\zeta=1}^{m} W(i)^2 U(i,r)^* U(i,k) \neq 0 \quad r = k
\]

\[
(4-10b) \quad \sum_{\zeta=1}^{m} W(i)^2 U(i,r)^* U(i,k) = 0 \quad r \neq k
\]

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If this is true, then \( [B] \) is a diagonal matrix and the following equation can be used to compute the enhanced frequency response function for mode \( r \).

\[
(4-11) \quad K(r,w) = \frac{\sum_{j=1}^{m} W(i)^z H(i,j,w) \ U(i,r)}{\sum_{i=1}^{n} W(i)^z U(i,r) \ U(i,r)}
\]

If it is not true, then it is necessary to solve simultaneously Equations 4-7 for the enhanced frequency response functions. In general, the weighting function is unknown. However, for the finite element case, the reduced mass or stiffness matrix can be used as the weighting function. For the case involving a modal vector estimate as the weighting function, unity weighting can be added based upon measurement point, direction, and/or component.

This technique is a powerful procedure for rapidly evaluating the correlation of a finite element model with the measured frequency response data taken during a ground vibration test. The finite element eigenvectors and a reduced mass matrix is used with the measured frequency response to compute the enhanced frequency response function.

\[
(4-12) \quad K(r,w) = \frac{\{U^*(r)\} [M] \{H(i,j,w)\}^T}{\{U^*(r)\} [M] \{U(r)\}^T}
\]
\( \mathbf{H_f} \) is a row vector of the values of the frequency responses for the points on the structures evaluated at a certain frequency. If the enhanced frequency response function has the form of a single degree of freedom, then \( \{U(r)\} \) must be an eigenvector of the system. Therefore, its corresponding complex pole can be estimated from the enhanced frequency response function. If the enhanced frequency response function is not a single degree of freedom function, then \( \{U(r)\} \) is not an eigenvector, the reduced mass matrix is incorrect, the frequency response measurements are in error, or a combination of the above. The advantage of the above technique is that it requires only a set of frequency response measurements and the finite element results. It is not necessary to determine the experimental modal parameters. If the finite element eigenvectors do not correlate, then the enhanced frequency response function can be used to obtain an estimate of the complex poles. The modal vectors can then be estimated using any modal parameter estimation scheme.

4.3 ENHANCED FREQUENCY RESPONSE FUNCTION APPLICATIONS

With respect to the frequency response function method of experimental modal analysis, the enhanced frequency response function procedure can be applied in at least two ways. First of all, this procedure can be used to determine the contribution of
a given modal vector to the frequency response function matrix. This technique can be very important in a ground vibration test on an aircraft because the modal vector to be checked can be the eigenvectors taken directly from the finite element model. This modal enhancement method is used to determine if the actual frequency response information taken from the airplane could possibly be constructed from the eigenvectors produced by the finite element model. This procedure is not in any way dependent upon the estimated frequency and damping values or the estimated modal vectors.

The second application of the enhanced frequency response function procedure is to actually enhance a particular mode of vibration so that better estimates of the frequency and damping values can be obtained from the experimental data. In this application, the finite element eigenvectors or the estimated modal vectors can be used as a weighting function to determine refined estimates of the global frequency and damping values which are in turn used to obtain better estimates of the modal vectors.

4.3.1 EIGENVECTOR WEIGHTING EXAMPLE

As an example of the enhanced frequency response function procedure, a finite element model was generated for the test
specimen of the T-plate. The eigenvectors for the first five
deflection modes were used as the modal vectors in the
enhancement procedure, as documented by Equation 4-11. Equation
4-12 was not used in this particular enhancement procedure since
a reduced mass matrix that would be required for the weighting
process, was not available. Even under this restriction the
results are quite favorable.

Figure 4-1 contains plots of two typical frequency response
functions obtained from measurements on the T-plate. For this
test, the frequency response measurements were made using an
accelerometer to measure the response to impact excitation at
every point of interest on the structure. Even in this simple
structure, it is possible that single degree of freedom modal
parameter estimation algorithms may have difficulty separating
the modal contributions of the two modes of vibration in the
region between 800 Hertz and 900 Hertz.

Figure 4-2 contains the enhanced frequency response functions for
the first five modes identified by the finite element model. It
should be noted that no experimental modal parameter estimation
has been required in order to generate any of these plots.
Figure 4-2a and Figure 4-2b confirm the frequency/damping
information that can be easily determined from either of the
typical frequency response functions shown in Figure 4-1.
Additionally, though, the fact that predominantly only one mode
Figure 4-1: Typical Frequency Response Functions - TPLATE
remains in the enhanced frequency response is a strong indication that the finite element model is generating information that is correlating with the experimental data in raw form. The inability of the modal enhancement procedure to generate a frequency curve approximating a single degree of freedom frequency response function in at least some frequency band is an indication of one of two problems. First, the frequency response function data may not contain the modal vector used in the enhancement. This will happen when the excitation or response is always located at a node of the modal vector used in the enhancement. Second, the finite element modal may be predicting a modal vector that is inappropriate for the actual structure. Figure 4-2c is an example of an enhanced frequency response function that is generated in this case. Since some enhancement is present in this enhanced frequency response function, the probable situation is that, for this data set, the response transducer is located at a node of the modal vector used in the enhancement. This was confirmed by knowledge of the eigenvector and of the test configuration.

Figure 4-2d and Figure 4-2e demonstrate another advantage of the enhanced frequency response functions. In this situation, the two closely spaced modes were very different in nature (torsion compared to bending). Even though no mass matrix was available for the weighting, the two modes are completely separated in Figure 4-2d and Figure 4-2e. This will allow single degree of
Figure 4-2: Enhanced Frequency Response Functions - TPLATE
Figure 4-2: Enhanced Frequency Response Functions - TPLATE
freedom modal parameter algorithms to be used to obtain accurate estimates of the frequency and damping of each mode independently.

4.3.2 MODAL VECTOR WEIGHTING EXAMPLE

As a second example of the enhanced frequency response function procedure, modal vectors were generated from a set of frequency response functions obtained on a Schweitzer sailplane. This data was generated using a fixed excitation point with a random excitation signal. The responses at every point of interest in the three orthogonal directions were measured by accelerometers. The modal vectors were generated by using the quadrature single degree of freedom modal parameter estimation technique. Figure 4-3 is an example of two typical frequency response functions obtained as part of the data set.

Figure 4-4 represents the enhanced frequency response functions for the first five modal vectors identified by the experimental modal parameter estimation process. In each case, a distinct reduction in the degrees of freedom in the neighborhood of the frequency of the modal vector used in the enhancement can be noted. Figure 4-4b and Figure 4-4c show the case of two closely spaced modes where the modal vectors are similar (bending compared to bending). While both modes are enhanced in both enhanced
Figure 4-3: Typical Frequency Response Functions - SAILPLANE
frequency response functions, clearly one mode is enhanced in preference to the other in each case and both cases show a clear improvement over the raw frequency response functions.

From each of the enhanced frequency response functions, improved estimates of the frequency and damping of selected system poles can be obtained. This information can now be utilized by a multiple degree of freedom modal parameter estimation algorithm in order to obtain improved estimates of the modal vectors. Obviously, this procedure can be repeated in an iterative fashion to determine if any changes in the modal parameter estimates occur.
Figure 4-4: Enhanced Frequency Response Functions - SAILPLANE
Figure 4-4: Enhanced Frequency Response Functions - SAILPLANE
REFERENCES


5.1 OVERVIEW

The common approach to estimation of modal vectors from the frequency response function method is to measure a complete row or column of the frequency response function matrix. This will give reasonable definition to those modal vectors that have a non-zero modal coefficient at the excitation location and can be completely uncoupled with the forced normal mode excitation method. When the modal coefficient at the excitation location of a modal vector is zero (very small with respect to the dynamic range of the modal vector) or when the modal vectors cannot be uncoupled, the estimation of the modal vector will contain potential bias and variance errors. In such cases additional rows and/or columns of the frequency response function matrix are measured to detect such potential problems.

Richardson and Kniskern [1] have suggested that a simple procedure of averaging two columns will reduce the variance error on the resulting estimate of the modal vector. Identification
and proper weighting of rows or columns containing poor estimates of a particular modal vector should greatly improve this process. Additionally, though, much more information concerning each modal vector used in such a procedure as well as an indication of the presence of bias errors in the estimate of the modal vector is desirable.

In general, this portion of the dissertation centers on developing practical techniques for the utilization of the redundant modal vector data in the frequency response function matrix. Specifically, information in the residue matrix corresponding to each pole of the system is evaluated in a least squares error approach to determine separate estimates of the same modal vector. This evaluation consists of the calculation of a complex modal scale factor (relating two modal vectors) and a scalar modal assurance criterion (measuring the consistency between two modal vectors).

The function of the modal scale factor (MSF) is to provide a means of normalizing all estimates of the same modal vector. When two modal vectors are scaled similarly, elements of each vector can be averaged (with or without weighting), differenced, or sorted to provide a best estimate of the modal vector or to provide an indication of the type of error vector superimposed on the modal vector.

The function of the modal assurance criterion (MAC) is to provide
a measure of consistency between estimates of a modal vector. This provides an additional confidence factor in the evaluation of a modal vector from different excitation locations. The modal assurance criterion also provides a method of determining the degree of causality between estimates of different modal vectors from the same system.

The modal scale factor and the modal assurance criterion also provide a method of easily comparing estimates of modal vectors originating from different sources. The modal vectors from a finite element analysis can be compared and contrasted with those determined experimentally as well as modal vectors determined by way of different experimental or modal parameter estimation methods. In this approach, methods can be compared and contrasted in order to evaluate the mutual consistency of different procedures rather than estimating the modal vectors specifically.

5.2 FREQUENCY RESPONSE FUNCTION THEORY - MODAL VECTORS

The formulation of the frequency response function matrix can be made in terms of the more general case of the transfer function. Therefore:
\[ (5-1) \quad \{X(s)\} = [H(s)] \{F(s)\} \]

where:

\( \{X(s)\} = \text{Response vector} \)
\( \{X(s)\} = m \times 1 \text{ column vector} \)
\( [H(s)] = \text{Transfer function matrix} \)
\( [H(s)] = m \times q \text{ rectangular matrix} \)
\( \{F(s)\} = \text{Excitation vector} \)
\( \{F(s)\} = q \times 1 \text{ column vector} \)
\( m = \text{Number of response stations} \)
\( q = \text{Number of excitation stations} \)
\( s = \text{LaPlace variable} \)
\( s = \sigma + j \omega \)

Each transfer function can now be written as a partial fraction expansion in terms of the system poles and residues. Assuming that the poles are global properties of the system yields:

\[ (5-2) \quad H(i,j,s) = \sum_{r=1}^{n} \frac{a(i,j,r)}{s - p(r)} + \frac{a^*(i,j,r)}{s - p^*(r)} \]

where:

\( a(i,j,r) = \text{Residue} \)
\( p(r) = \text{System pole for mode } r \)
\( * = \text{Complex conjugate} \)
\( i = \text{Row index} \)
\( i = \text{Response location} \)
\( j = \text{Column index} \)
\[ j = \text{Excitation location} \]
\[ r = \text{Mode number} \]
\[ n = \text{Number of degrees of freedom} \]

Rewriting Equation 5-2 in matrix form to correspond to Equation 5-1 gives:

\[
(5-3) \quad [H(s)] = \sum_{r=1}^{M} \frac{[A(r)]}{s - p(r)} + \frac{[A^*(r)]}{s - p^*(r)}
\]

where:
\[ [A(r)] = \text{Residue matrix} \]
\[ [A^*(r)] = m \times q \text{ rectangular matrix} \]

Furthermore, each element of the residue matrix of Equation 5-3 can be written:

\[
(5-4) \quad a(i,j,r) = k(r) \times U(i,r) \times U(j,r)
\]

where:
\[ k(r) = \text{Scaling constant for mode } r \]
\[ U(i,r) = \text{modal coefficient for location } i \text{ of mode } r \]
\[ U(j,r) = \text{modal coefficient for location } j \text{ of mode } r \]

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In terms of matrix notation:

\[
\begin{bmatrix}
U(1,r)U(1,r) & U(1,r)U(2,r) & U(1,r)U(3,r) \\
U(2,r)U(1,r) & U(2,r)U(2,r) & U(2,r)U(3,r) \\
U(3,r)U(1,r) & U(3,r)U(2,r) & \cdots \\
U(4,r)U(1,r) & U(4,r)U(2,r) & \cdots \\
U(5,r)U(1,r) & \cdots & \cdots \\
U(6,r)U(1,r) & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{bmatrix}
\]

(5-5) \quad [A(r)] = k(r)

Equations 5-4 and 5-5 state that each element of the residue matrix for modal vector \( r \) consists of the product of a scaling constant, the modal coefficient of the excitation location, and the modal coefficient of the response location. Equation 5-5 can be restated in a more concise matrix form as follows:

\[
[\mathbf{A}(r)] = k(\mathbf{r}) \{U(\mathbf{r})\} \{U(\mathbf{r})\}^T
\]
From Equation 5-6 it is easy to see that each row or column is simply the same modal vector multiplied times the modal coefficient of the common response location or excitation location respectively. Therefore, the proportionality constant between rows C and D or columns C and D can be referred to as a modal scale factor equal to:

\[(5-7) \quad \text{MSF}(c,d,r) = \frac{U(c,r)}{U(d,r)}\]

While the formulation of Equation 5-7 is made without designation of row or column, the application of the concept of modal scale factor will differ according to whether the modal scale factor is determined from row or column relationships. This will be discussed further in Section 5.6, Modal Assurance Criterion Applications.

5.3 ORTHOGONALITY OF MODAL VECTORS

Theoretically, for the case of proportional damping, each modal vector of a system will be orthogonal to all other modal vectors of that system when weighted by the mass, stiffness, or damping matrix. In practice, these matrices are made available by way of

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a finite element analysis and normally the mass matrix is considered to be the most accurate. For this reason, any further discussion of orthogonality will be made with respect to mass matrix weighting. As a result, the orthogonality relations can be stated as follows:

\[(5-8) \quad \{U(r)\}^T \begin{bmatrix} M \end{bmatrix} \{U(q)\} = 0 \quad r \neq q\]

\[(5-9) \quad \{U(r)\}^T \begin{bmatrix} M \end{bmatrix} \{U(q)\} = M(r) \quad r = q\]

where: \[M(r) = \text{Generalized mass of mode } r\]

Experimentally, the result of zero in Equation 5-8 can rarely be achieved but values up to one tenth of the magnitude of the generalized mass of each mode are considered to be acceptable. It is common procedure to form the modal vectors into a normalized set of mode shape vectors with respect to the mass matrix weighting. The accepted criterion in the aerospace industry, where this confidence check is made most often, is for all of the generalized mass terms to be unity and all terms resulting from Equation 5-8 to be less than 0.1. Often, even under this criteria, an attempt is made to adjust the modal vectors so that Equations 5-8 and 5-9 are satisfied \[2,3,4\].
In Equations 5-8 and 5-9 the mass matrix must be an $m \times m$ matrix corresponding to the measurement locations on the structure. This means that the finite element mass matrix must be modified from whatever size and distribution of grid locations required in the finite element analysis to the $m \times m$ square matrix corresponding to the measurement locations. This normally involves some sort of reduction algorithm as well as interpolation of grid locations to match the measurement situation [5,6].

When Equation 5-8 is not sufficiently satisfied, one (or more) of three situations may exist. First, the modal vectors can be invalid. This can be due to measurement error or problems with the modal parameter estimation algorithms. This is a very common assumption and many times contributes to the problem. Second, the mass matrix can be invalid. Since the mass matrix is not easily related to the physical properties of the system, this probably contributes significantly to the problem. Third, the reduction of the mass matrix can be invalid [5-8]. This can certainly be a realistic problem and cause severe errors. The most obvious example of this situation would be when a relatively large amount of mass is reduced to a measurement location that is highly flexible, such as the center of an unsupported panel. In such a situation the measurement location is weighted very heavily in the orthogonality calculation of Equation 5-8 but may represent only incidental motion of the overall modal vector.
In all probability, all three situations contribute to the failure of Equation 5-8 to be satisfied on occasion. When Equation 5-8 is not satisfied, this result does not indicate where the problem originates. From an experimental point of view, it is important to try to develop methods that indicate confidence that the modal vector is or is not part of the problem.

Since the residue matrix contains redundant information with respect to a modal vector, the consistency of the estimate of the modal vector under varying conditions such as excitation location or modal parameter estimation algorithms can be a valuable confidence factor to be utilized in the process of evaluation of the experimental modal vectors.

5.4 CONSISTENCY OF MODAL VECTORS

Although it has long been known that the frequency response matrix (and therefore the residue matrix) contains redundant information, little use of this fact has been made. In fact, the primary use of this information has been to justify measuring the minimum number of rows or columns to obtain a complete set of modal vectors. The primary reason for measuring the minimum
amount of the matrix has been the time required to obtain and process the data. With the increasing availability of digital test equipment, parallel data processing and parallel data analysis, the time constraint is no longer as severe. Since it is often not possible to excite all modal vectors of interest from a single excitation location, a minimum of two or three rows or columns will have to be measured. Presently, the redundant information obtained in this manner is examined visually to evaluate discrepancies and determine which modal vector represents the best estimate. As more rows and/or columns are involved, this approach cannot fully take advantage of the redundant information efficiently.

Ideally, all data in each residue matrix should be evaluated, weighted as to importance, and utilized in some manner, manual or automatic, in order to determine the best estimate of the modal vector or the source of the contamination of the modal vector. In order to proceed along these lines, a correlation coefficient between modal vector estimates would be very beneficial.

5.4.1 MODAL ASSURANCE CRITERION

If Equation 5-7 is used as the basis for a model to calculate a least squares error estimate of the proportionality constant
between rows or columns of the residue matrix, the model is linear as follows:

\[(5-10) \quad a(c, j, r) = \text{MSF}(c, d) \cdot a(d, j, r)\]

In vector notation this would be:

\[(5-11) \quad \{A(c, r)\} = \text{MSF}(c, d) \cdot \{A(d, r)\}\]

All of the elements of the modal vectors in rows/columns c and d exhibit the relationship stated in Equation 5-11. The value of the modal scale factor is to be calculated so as to minimize the sum of the squared errors between corresponding elements of each modal vector. All or part of each modal vector can be used in such a calculation. Obviously, if some elements are consciously excluded a form of weighted least squares error estimation is involved. The complete least squares developement is shown in Appendix C. The modal scale factor is defined, according to this approach, as follows:

\[(5-12) \quad \text{MSF}(c, d) = \frac{\sum_{j=1}^{m} a(c, j, r) \cdot \overline{a(d, j, r)}}{\sum_{j=1}^{m} a(d, j, r) \cdot \overline{a(d, j, r)}}\]

The numerator of Equation 5-12 can be defined as the cross moment
of the modal vectors. This will be represented by:

\[(5-13) \quad \text{MOM}(c,d) = \sum_{j=1}^{m} a(c,j,r) \ast a(d,j,r)\]

Similarly, the denominator of Equation 5-12 can be defined as the auto moment of the modal vectors. This will be represented by:

\[(5-14) \quad \text{MOM}(d,d) = \sum_{j=1}^{m} a(d,j,r) \ast a(d,j,r)\]

Therefore, Equation 5-12 can be restated in a more concise manner

\[(5-15) \quad \text{MSF}(c,d) = \frac{\text{MOM}(c,d)}{\text{MOM}(d,d)}\]

Equation 5-15 implies that the modal vector of row/column D is the reference to which the modal vector of row/column C is compared. In the general case, modal vector C can be considered to be made of two parts. The first part will be the part correlated with modal vector D. The second part will be the part that is not correlated with modal vector D and will be made up of contamination from other modal vectors and of any random contribution. This error vector will be considered to be noise. If the modal assurance criterion is defined as a scalar constant relating the portion of the auto moment of the modal vector that
is linearly related to the reference modal vector, then the following equation is applicable:

\[(5-16) \quad \text{MAC}(c,d) \cdot \text{MOM}(c,c) = \left| \text{MSF}(c,d) \right|^2 \cdot \text{MOM}(d,d)\]

Therefore, solving for the modal assurance criterion:

\[(5-17) \quad \text{MAC}(c,d) = \frac{\left| \text{MSF}(c,d) \right|^2 \cdot \text{MOM}(d,d)}{\text{MOM}(c,c)}\]

Substituting Equation 5-15 into Equation 5-17 gives the final form of the equation defining the modal assurance criterion.

\[(5-18) \quad \text{MAC}(c,d) = \frac{\left| \text{MOM}(c,d) \right|^2}{\text{MOM}(c,c) \cdot \text{MOM}(d,d)}\]

The modal assurance criterion is a scalar constant relating the causal relationship between two modal vectors. The constant will take on values from zero, representing no consistent correspondence, to one, representing a consistent correspondence. In this manner, if the modal vectors under consideration truly exhibit a consistent relationship, the modal assurance criterion should approach unity and the value of the modal scale factor can be considered to be reasonable.
5.4.1.1 FREQUENCY RESPONSE FUNCTION ANALOGY

The formulation of the frequency response function and coherence at a specific frequency is a direct parallel to the concept of the modal scale factor and modal assurance criterion. The common approach to the formulation of the frequency response and coherence functions is based upon the following linear model:

\[ (5-19) \quad Y = HX + N \]

This model is used in a least squares error formulation in order to estimate the frequency response and coherence functions just as the least squares approach is used to define the modal scale factor and the modal assurance criterion. Instead of moments between modal vectors, the moments are calculated between input and output spectrums. Assuming that the noise is not correlated with the input, the estimation of the frequency response function \( H(y,x) \) for an input \( X \) and an output \( Y \) is:

\[ (5-20) \quad H(y,x) = \frac{G(y,x)}{G(x,x)} \]

where: \[ G(y,x) \] = Cross spectrum between output and input spectrum
\[ G(x,x) \] = Auto spectrum of input
The coherence function is a scalar quantity indicating the causal relationship between the output Y and the input X. It is likewise calculated in a manner corresponding to the modal assurance criterion.

\[(5-21) \quad \text{COH}(y,x) = \frac{\| G(y,x) \|^2}{G(x,x) G(y,y)} \]

where:
- \( G(y,y) = \) Auto spectrum of output
- \( \text{COH}(y,x) = \) Coherence function

The reason for noting this parallel analogy is that much analysis has been done in the area of frequency response and coherence functions with respect to noise contamination and error evaluation. Therefore, these equivalent concepts will provide understanding of the applications and limitations of the modal scale factor and the modal assurance criterion.

5.4.1.2 LINEAR REGRESSION ANALOGY

The development of the relationship between two random variables X and Y can be formulated on the basis of linear regression techniques. In this case, the same linear model can be used as in Equation 5-19. This is restated here as:
(5-22) \[ Y = H X + N \]

For this situation, the moments are calculated on the basis of covariance functions between the variables \( X \) and \( Y \). Therefore, the constant \( H \) can be formulated as

(5-23) \[ H(y,x) = \frac{\text{COV}(y,x)}{\text{COV}(x,x)} \]

where:

- \( \text{COV}(y,x) = \text{Covariance between } Y \text{ and } X \)
- \( \text{COV}(y,x) = \mathbb{E}[(y - \bar{y})(x - \bar{x})] \)
- \( \text{COV}(x,x) = \text{Square of the standard deviation of } X \)
- \( \text{COV}(x,x) = \text{Variance of } X \)

As a companion to the modal assurance criterion, a correlation coefficient can be formulated on the same basis.

(5-24) \[ \text{COR}(y,x) = \frac{1}{\text{COV}(x,x) \text{ COV}(y,y)} \left| \frac{\text{COV}(y,x)}{\text{COV}(y,y)} \right|^2 \]

where:

- \( \text{COV}(y,y) = \text{Square of the standard deviation of } Y \)
- \( \text{COV}(y,y) = \text{Variance of } Y \)
- \( \text{COR}(y,x) = \text{Square of the correlation} \)
Likewise, analysis concepts applicable to the linear regression approach can provide insight as to potential ramifications of the modal scale factor and the modal assurance criterion.

5.5 MODAL ASSURANCE CRITERION PROPERTIES

The value of the modal assurance criterion can give an indication as to the validity of the modal scale factor. While certain implications of the modal assurance criterion are dependent upon the calculations involving rows or columns, some general discussion is applicable to all cases.

The modal assurance criterion can take on values between zero and one. If the modal assurance criterion has a value near zero, this is an indication that the modal vectors are not consistent. This can be due to any of the following reasons:

1) The system is non-stationary. This can occur whenever the system is undergoing a change in mass or stiffness
during the testing period.

2) The system is nonlinear. System nonlinearities will appear differently in frequency response functions generated from different exciter positions or excitation signals. The modal parameter estimation algorithms will also not handle the different nonlinear characteristics in a consistent manner.

3) There is noise on the reference modal vector. This case is the same as noise on the input of a frequency response function measurement. No amount of signal processing can remove this type of error.

4) The modal parameter estimation is invalid. The frequency response functions measurements may contain no errors but the modal parameter estimation may not be consistent with the data. For example, the modal parameter estimation algorithm may utilize a complex system pole model when only real valued system poles exist.

5) The modal vectors are from linearly unrelated mode shape vectors. Hopefully, since the different modal vector estimates are from different excitation positions this measure of inconsistency will imply that the modal vectors are orthogonal.

Obviously, if the first four reasons can be eliminated, the modal
assurance criterion can be interpreted in a similar way as an orthogonality calculation.

If the modal assurance criterion has a value near unity, this is an indication that the modal vectors are consistent. This does not necessarily mean that they are correct. The modal vectors can be consistent for any of the following reasons:

1) The modal vectors have been incompletely measured. This situation can occur whenever too few response stations have been included in the experimental determination of the modal vector. For example, two symmetric modes of the wings of an airplane will appear to be identical if only response stations at the wing tips are used to define the modal vectors.

2) The modal vectors are the result of a forced excitation other than the desired input. This would be the situation if, during the measurement of the frequency response function, a rotating piece of equipment with an unbalance is present in the system being tested.

3) The modal vectors are primarily coherent noise. Since the reference modal vector may be arbitrarily chosen, this modal vector may not be one of the true modal vectors of the system. It could simply be a random noise vector or a vector reflecting the bias in the modal parameter estimation.
algorithm. In any case, the modal assurance criterion will only reflect a causal relationship to the reference modal vector.

4) The modal vectors represent the same modal vector with different arbitrary scaling. If the two modal vectors being compared have the same expected value when normalized, the two modal vectors should differ only by the complex valued scale factor which is a function of the common modal coefficients between the rows or columns.

Therefore, if the first three reasons can be eliminated, the modal assurance criterion indicates that the modal scale factor is the complex constant relating the modal vectors and that the modal scale factor can be used to average, difference, or sort the modal vectors.

It is very important to notice that the modal assurance criterion can only indicate consistency, not validity. If the same errors, random or bias, exist in all modal vector estimates, this will not be delineated by the modal assurance criterion. Invalid assumptions are normally the cause of this sort of potential error. Even though the modal assurance criterion is unity, the assumptions involving the system or the modal parameter estimation techniques are not necessarily correct. The assumptions may cause consistent errors in all modal vectors under all test conditions verified by the modal assurance
criterion.

5.6 MODAL ASSURANCE CRITERION APPLICATIONS

Under the constraints mentioned in the previous section, the modal assurance criterion can be applied in many different ways. The modal assurance criterion can be used to verify or correlate an experimental modal vector with respect to a theoretical modal vector (eigenvector). Experimental modal vectors can be averaged or differenced to determine the best single estimate or the potential source of contamination. The modal assurance criterion can be calculated based upon common elements of incomplete modal vectors in order to form a complete modal vector. The modal assurance criterion can be used to evaluate modal parameter estimation methods if a set of theoretical modal vectors is used in common to a variety of methods. Finally, the modal assurance criterion can be utilized in an experimental test procedure to determine the modal vectors without knowing the explicit details of the input. The following sections will discuss some of these applications in detail.
5.6.1 VERIFICATION OF MODAL VECTORS

One possible method of verifying experimental modal vectors is to make a comparison of the experimental modal vectors to a set of eigenvectors computed from a finite element program. The assumption in this case is that the eigenvectors are representative of the actual modal vectors of the system. The following symmetric matrices represent such a comparison for the T-plate shown in Figure 5-1.

In each of the two matrices in Figure 5-2, the modal assurance criterion has been formulated between all possible combinations of modal vectors found by each method. In both cases, the modal assurance criterion matrix indicates that there is no linear relationship among any of the modal vectors found within each set.

These two sets of modal vectors are now compared in the nonsymmetric matrix in Figure 5-3. For this case, the row of the matrix represents each of the nine experimental modal vectors and the column of the matrix represents each of the ten finite element eigenvectors. The matrix includes only those elements of each modal vector for which frequency response function information was actually acquired. The results of this modal assurance criterion matrix are tabulated in Figure 5-4. This
### Modal Assurance Criterion: TPL

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Figure 5-2: Modal Assurance Criterion Matrix - TPLATE
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Figure 5-3: Modal Assurance Criterion Matrix - TPLATE
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Figure 5-4: Modal Vector Correlation - TPLATE

The figure indicates that two of the finite element eigenvectors were not found from this set of experimental data. Both of these modal vectors were not identified since a node of each modal
vector was located at the point which remained fixed during the frequency response function experimental modal analysis procedure.

5.6.2 EVALUATION OF MODAL VECTORS

The evaluation of modal vectors can be greatly facilitated through the use of the modal assurance criterion. One problem that is often encountered is the reduction of many different modal vectors and many different estimates of the same modal vector, obtained from a number of excitation locations, to a set representing the best estimate for each modal vector. This problem was investigated through the use of seven different excitation configurations on the Schweitzer sailplane represented in Figure 5-5. The excitation locations are identified in Figure 5-6.

Figure 5-7 through Figure 5-13 summarize the results of each excitation location. Each of the modal assurance criterion matrices were formulated based upon modal vectors found by using the quadrature single degree of freedom method of modal parameter estimation. Within these matrices, modal assurance criterion values that appear to be significant need to be investigated further. For example, a multiple degree of freedom modal parameter estimation algorithm may be required in order to reduce
Figure 5-5: SAILPLANE Representation
Figure 5-5: SAILPLANE Representation
TEST IDENTIFICATION: SAILPLANE1

EXCITATION POINT: 1
EXCITATION DIRECTION: +2
EXCITATION SIGNAL: PURE RANDOM
BOUNDARY CONDITION: LANDING WHEEL

DAMPED NATURAL FREQUENCIES:

| MODAL VECTOR 1     | 4.9 Hertz  |
| MODAL VECTOR 2     | 8.9 Hertz  |
| MODAL VECTOR 3     | 10.8 Hertz |
| MODAL VECTOR 4     | 12.8 Hertz |
| MODAL VECTOR 5     | 24.1 Hertz |
| MODAL VECTOR 6     | 27.4 Hertz |
| MODAL VECTOR 7     | 29.1 Hertz |
| MODAL VECTOR 8     | 31.0 Hertz |
| MODAL VECTOR 9     | 37.9 Hertz |

MODAL ASSURANCE CRITERION:

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Figure 5-7: Modal Assurance Criterion Matrix - SAILPLANE1

172
**TEST IDENTIFICATION:** SAILPLANE2

**EXCITATION POINT:** 1
**EXCITATION DIRECTION:** +Z
**EXCITATION SIGNAL:** PURE RANDOM
**BOUNDARY CONDITION:** FREE-FREE

**DAMPED NATURAL FREQUENCIES:**

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<td>20.7 Hertz</td>
<td>24.0 Hertz</td>
<td>30.4 Hertz</td>
<td>31.0 Hertz</td>
<td>38.0 Hertz</td>
</tr>
</tbody>
</table>

**MODAL ASSURANCE CRITERION:**

\[
\begin{array}{cccccccc}
1.000 & 0.000 & 0.001 & 0.055 & 0.013 & 0.002 & 0.025 & 0.001 & 0.007 \\
1.000 & 0.046 & 0.031 & 0.014 & 0.130 & 0.001 & 0.037 & 0.002 \\
1.000 & 0.005 & 0.000 & 0.114 & 0.000 & 0.020 & 0.007 \\
1.000 & 0.005 & 0.000 & 0.243 & 0.016 & 0.046 \\
1.000 & 0.001 & 0.003 & 0.009 & 0.007 \\
1.000 & 0.022 & 0.007 & 0.003 \\
1.000 & 0.244 & 0.003 \\
1.000 & 0.015 \\
1.000 \\
\end{array}
\]

*Figure 5-8: Modal Assurance Criterion Matrix - SAILPLANE2*
TEST IDENTIFICATION: SAILPLANE3

EXCITATION POINT: 89
EXCITATION DIRECTION: +Z
EXCITATION SIGNAL: RANDOM TRANSIENT
BOUNDARY CONDITION: FREE-FREE

DAMPED NATURAL FREQUENCIES:

<table>
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<tr>
<th>MODAL VECTOR 1</th>
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<th>MODAL VECTOR 3</th>
<th>MODAL VECTOR 4</th>
<th>MODAL VECTOR 5</th>
<th>MODAL VECTOR 6</th>
<th>MODAL VECTOR 7</th>
<th>MODAL VECTOR 8</th>
<th>MODAL VECTOR 9</th>
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<td>4.2 HERTZ</td>
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<td>9.8 HERTZ</td>
<td>12.6 HERTZ</td>
<td>13.7 HERTZ</td>
<td>24.6 HERTZ</td>
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<td>32.7 HERTZ</td>
<td>38.1 HERTZ</td>
</tr>
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</table>

MODAL ASSURANCE CRITERION:

\[
\begin{array}{cccccccccc}
1.000 & 0.014 & 0.007 & 0.003 & 0.425 & 0.047 & 0.010 & 0.050 & 0.020 \\
1.000 & 0.024 & 0.006 & 0.026 & 0.011 & 0.064 & 0.000 & 0.013 \\
1.000 & 0.000 & 0.005 & 0.000 & 0.084 & 0.000 & 0.001 \\
1.000 & 0.000 & 0.029 & 0.052 & 0.010 & 0.251 \\
1.000 & 0.000 & 0.003 & 0.023 & 0.042 \\
1.000 & 0.006 & 0.070 & 0.010 \\
1.000 & 0.142 & 0.049 \\
1.000 & 0.013 \\
1.000
\end{array}
\]

Figure 5-9: Modal Assurance Criterion Matrix - SAILPLANE3
TEST IDENTIFICATION: SAILPLANE4

EXCITATION POINT: 22
EXCITATION DIRECTION: +Z
EXCITATION SIGNAL: PURE RANDOM
BOUNDARY CONDITION: FREE-FREE

DAMPED NATURAL FREQUENCIES:

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<th>MODAL VECTOR 3</th>
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<th>MODAL VECTOR 5</th>
<th>MODAL VECTOR 6</th>
<th>MODAL VECTOR 7</th>
<th>MODAL VECTOR 8</th>
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<td>9.8 HERTZ</td>
<td>12.6 HERTZ</td>
<td>13.7 HERTZ</td>
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<td>38.0 HERTZ</td>
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MODAL ASSURANCE CRITERION:

\[
\begin{array}{cccccccccc}
1.000 & 0.001 & 0.001 & 0.000 & 0.212 & 0.002 & 0.153 & 0.013 & 0.025 \\
1.000 & 0.042 & 0.033 & 0.005 & 0.194 & 0.006 & 0.023 & 0.000 \\
1.000 & 0.033 & 0.007 & 0.095 & 0.001 & 0.000 & 0.009 \\
1.000 & 0.008 & 0.001 & 0.001 & 0.008 & 0.015 \\
1.000 & 0.002 & 0.229 & 0.006 & 0.034 \\
1.000 & 0.007 & 0.001 & 0.012 \\
1.000 & 0.067 & 0.007 \\
1.000 & 0.017 \\
1.000 & \\
\end{array}
\]

Figure 5-10: Modal Assurance Criterion Matrix - SAILPLANE4

175
TEST IDENTIFICATION: SAILPLANE5

EXCITATION POINT: 53
EXCITATION DIRECTION: -\(\gamma\)
EXCITATION SIGNAL: PURE RANDOM
BOUNDARY CONDITION: FREE-FREE

DAMPED NATURAL FREQUENCIES:

| MODAL VECTOR 1 | 8.7 Hertz |
| MODAL VECTOR 2 | 9.8 Hertz |
| MODAL VECTOR 3 | 18.9 Hertz |
| MODAL VECTOR 4 | 20.6 Hertz |
| MODAL VECTOR 5 | 21.3 Hertz |
| MODAL VECTOR 6 | 31.3 Hertz |
| MODAL VECTOR 7 | 38.1 Hertz |
| MODAL VECTOR 8 | 39.5 Hertz |

MODAL ASSURANCE CRITERION:

\[
\begin{bmatrix}
1.000 & 0.028 & 0.034 & 0.021 & 0.037 & 0.076 & 0.020 & 0.129 \\
1.000 & 0.036 & 0.017 & 0.011 & 0.040 & 0.001 & 0.078 \\
1.000 & 0.012 & 0.242 & 0.010 & 0.007 & 0.049 \\
1.000 & 0.178 & 0.005 & 0.129 & 0.290 \\
1.000 & 0.000 & 0.089 & 0.021 \\
1.000 & 0.035 & 0.026 \\
1.000 & 0.000 \\
1.000
\end{bmatrix}
\]

Figure 5-11: Modal Assurance Criterion Matrix - SAILPLANE5
TEST IDENTIFICATION: SAILPLANE6

EXCITATION POINT: 64
EXCITATION DIRECTION: -Y
EXCITATION SIGNAL: PURE RANDOM
BOUNDARY CONDITION: FREE-FREE

DAMPED NATURAL FREQUENCIES:

| MODAL VECTOR 1 | 4.3 HERTZ |
| MODAL VECTOR 2 | 8.8 HERTZ |
| MODAL VECTOR 3 | 16.7 HERTZ |
| MODAL VECTOR 4 | 19.2 HERTZ |
| MODAL VECTOR 5 | 21.4 HERTZ |
| MODAL VECTOR 6 | 24.3 HERTZ |
| MODAL VECTOR 7 | 31.2 HERTZ |
| MODAL VECTOR 8 | 38.1 HERTZ |
| MODAL VECTOR 9 | 39.7 HERTZ |

MODAL ASSURANCE CRITERION:

\[
\begin{array}{cccccccccc}
1.000 & 0.214 & 0.050 & 0.000 & 0.044 & 0.025 & 0.000 & 0.024 & 0.002 \\
1.000 & 0.062 & 0.017 & 0.022 & 0.195 & 0.064 & 0.067 & 0.020 \\
1.000 & 0.110 & 0.037 & 0.006 & 0.020 & 0.003 & 0.004 \\
1.000 & 0.166 & 0.006 & 0.011 & 0.003 & 0.153 \\
1.000 & 0.047 & 0.002 & 0.046 & 0.007 \\
1.000 & 0.000 & 0.001 & 0.001 \\
1.000 & 0.399 & 0.019 \\
1.000 & 0.274 \\
1.000 & \\
\end{array}
\]

Figure 5-12: Modal Assurance Criterion Matrix - SAILPLANE6
TEST IDENTIFICATION: SAILPLANE7

EXCITATION POINT: 33
EXCITATION DIRECTION: +Z
EXCITATION SIGNAL: PURE RANDOM
BOUNDARY CONDITION: FREE-FREE

DAMPED NATURAL FREQUENCIES:

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<tr>
<td>2</td>
<td>8.8 HERTZ</td>
</tr>
<tr>
<td>3</td>
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<tr>
<td>6</td>
<td>23.9 HERTZ</td>
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<tr>
<td>7</td>
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</tr>
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MODAL ASSURANCE CRITERION:

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</tr>
</tbody>
</table>

Figure 5-13: Modal Assurance Criterion Matrix - SAILPLANE7
the bias of one modal vector from another.

Using the information from these modal assurance criterion matrices, Figure 5-14 through Figure 5-16 can be formulated to compare the modal vectors, found from each excitation location, that appear, based upon frequency, to be associated with the same modal vector. In each of these figures, the modal assurance criterion computations were made for two cases. The first case includes all measurement points on the sailplane. The second case includes only those measurement points on the wings of the sailplane. Since most of the lower frequency modal vectors primarily consist of wing motion, the second case will weight this data more heavily. If no entry appears in a particular column or row of the modal assurance criterion matrix, then that modal vector could not be found from that excitation location. The information represented in Figure 5-14 through Figure 5-16 can be used to decide which modal vectors are to be included in an averaged modal vector if the correlation is good or which modal vector by itself represents the best estimate among the set of estimates of modal vectors.

5.6.3 COMBINATION OF MODAL VECTORS

From information like that generated by the previous example, modal vectors can be combined in a weighted summation to
### Modal Assurance Criterion: SAIL-4.3

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### Modal Assurance Criterion: SAIL-4.3 (Wings Only)

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**Figure 5-14:** Modal Assurance Criterion Matrix - SAILPLANE 4.3 Hertz Modal Vectors

180
MODAL ASSURANCE CRITERION:  SAIL-8.7

<table>
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MODAL ASSURANCE CRITERION:  SAIL-8.7 (WINGS ONLY)

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</table>

Figure 5-15: Modal Assurance Criterion - SAILPLANE 8.7 Hertz Modal Vectors
### Modal Assurance Criterion: SAIL-13.6

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</table>

### Modal Assurance Criterion: SAIL-13.6 (Wings Only)

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</tbody>
</table>

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**Figure 5-16:** Modal Assurance Criterion - SAILPLANE 13.6 Hertz Modal Vectors

182
determine similarities and differences. The first example of
this use of modal scale factor and modal assurance criterion was
formulated using data from the T-plate. The modal vector
occurring at 268.5 Hertz was estimated using data from two
different exciter positions. These two modal vectors are plotted
in Figure 5-17. Although these modal vectors appear to be very
similar, the modal assurance criterion formulated from these data
sets was not unity. Using the modal scale factor to normalize
the two modal vectors, the resultant modal vectors were
differenced with the result plotted in Figure 5-18. This plot
indicates that the only difference between these two estimates of
the same modal vector is random in distribution.

In a similar approach, two estimates of the same modal vector for
the sailplane were compared. The 4.3 Hertz mode is the first
symmetric bending mode of the wings of the sailplane. Based upon
the modal assurance criterion data from Figure 5-14, a weighted
summation of modal vectors from the first, second, fourth, and
seventh modal tests was formed. This modal vector is plotted in
Figure 5-19. The modal vector at the same frequency from the
third modal test is plotted in Figure 5-20. Both of these modal
vectors appear to be very similar but the modal assurance
criterion indicates that there is a substantial difference.
Using the modal scale factor to once again normalize these two
modal vectors, the resulting difference is plotted in Figure
5-21. This figure clearly indicates that the contamination of
Figure 5-17: Separate Estimates of a Modal Vector - TPLATE
Figure 5-18: Difference Between Normalized Estimates of Modal Vectors - TPLATE
Figure 5-19: Weighted Summation of Modal Vectors - SAILPLANE
Figure 5-20: Modal Vector from Third SAILPLANE Modal Test
Figure 5-21: Difference Between Normalized Estimates of Modal Vectors - SAILPLANE
the modal vector from the third modal test is a bias error originating from the second symmetric bending mode of the wings of the sailplane. This bias error in the modal vector could possibly be reduced through the use of multiple degree of freedom modal parameter estimation algorithms.

5.6.4 EVALUATION OF MODAL TEST METHODOLOGY

The evaluation of modal test parameters can be accomplished by comparing data set(s) originating from the same test structure. The comparison of the data sets from the seven excitation positions on the sailplane is one example of this kind of analysis. Another example is the evaluation of modal test methodology is to compare different modal parameter estimation algorithms based upon the same set of data. Figure 5-22 through Figure 5-24 represent the comparison of modal parameter estimation algorithms for the T-plate and the sailplane. In each case the following modal parameter estimation algorithms were used to estimate the modal vector associated with the same row/column index:

1) Amplitude - Single Degree of Freedom
2) Quadrature - Single Degree of Freedom
3) Circle Fit - Single Degree of Freedom
4) Least Squares Complex Exponential - Multiple Degree of
Freedom

Approximately the same amount of time was used in the modal parameter estimation for each of the algorithms. In the case of the circle fit algorithm, this required the modal parameter estimation to proceed in an automatic manner. This is not recommended for the circle fit algorithm and is the probable reason for extremely poor correlation of the circle fit method with the other methods.

5.6.5 DETERMINATION OF MODAL VECTORS

The response ratio method is a technique that uses the concept of modal scale factor and modal assurance criterion, computed on the basis of row relationships, in order to determine modal vectors. An implicit knowledge that the excitation spectrum contains no poles or zeros in the frequency range of interest is required although no explicit knowledge is needed.

Unlike the frequency response function method, in which the residue could be used directly as a measure of a modal vector element, the residue of the response function is arbitrary due to its dependence on the input spectrum. In order to scale the response residues, it is necessary to measure the response function at a minimum of two points on the structure. One point is a reference point that is common to all measurements for a
**MODAL ASSURANCE CRITERION:** TP1 853.2 Hertz

1.000 0.979 0.857 0.949  
1.000 0.930 0.990  
1.000 0.965  
1.000

**MODAL ASSURANCE CRITERION:** TP1 882.4 Hertz

1.000 0.993 0.997 0.994  
1.000 0.998 0.999  
1.000 0.999  
1.000

*Figure 5-22: Modal Assurance Criterion Matrix - TPLATE Modal Parameter Estimation Comparisons*
### MODAL ASSURANCE CRITERION: SAILPLANE2 8.6 Hertz

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### MODAL ASSURANCE CRITERION: SAILPLANE2 9.8 Hertz

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**Figure 5-23:** Modal Assurance Criterion - SAILPLANE
Modal Parameter Estimation Comparisons
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</table>

Figure 5-24: Modal Assurance Criterion Matrix - SAILPLANE
Modal Parameter Estimation Comparisons
modal vector, and the other point is the roving or indexed point of the modal vector. The ratio of the response residues for a given pole is equal to the ratio of the modal vector elements for these two points. Furthermore, the ratio of the modal vector elements, for a pole distinct from all others in the system, is unique. Consequently, the ratio of the response residues is unique.

If a repeated pole (a pole having multiplicity greater than unity) occurs, then the ratios of residues for the pole need not be unique. The residue associated with a multiple pole is a multilinear function of the elements corresponding to the excitation point in the several mode shapes for that pole, rather than a linear function. Therefore, variation in the excitation point will generally result in variation of the residue ratio.

To determine the modal vectors of the structure, a reference transducer position is chosen, and the responses to some forcing vector are measured simultaneously at the reference point and at a roving point selected on the structure. The ratio of the residue of the roving point with respect to the residue of the reference point is a measure of the modal vector element at the roving point. Using the same reference point, the process can be repeated for the next choice of a roving point on the structure. This process is repeated until all points on the structure have been measured. For each response ratio function, the response is
measured at the reference point and the roving point for a large number of forcing conditions.

If a good reference location is chosen (such that good estimates of the reference residue can be made) for a given mode, then despite poor estimates of the roving residue, good estimates of the modal vector can be obtained. If a poor reference point is picked, then biased results are obtained.

There is a significant advantage when using this response ratio method when a large number of response measurements can be taken simultaneously. For this case, any one of the set of response measurements can be used as a reference. This means that an optimum reference point, from among the points measured, can be found for each pole. Of course, this requires more extensive data acquisition equipment.

In the following sections, two examples will be given for determining modal information from response ratio data. The procedure that is illustrated in both of these examples is meant to demonstrate the characteristics and usefulness of the modal scale factor and modal assurance criterion for determining system dynamic characteristics.

5.6.5.1 DISTINCT POLE EXAMPLE

The first example is a complete modal survey of a standard test
item. This item is a T-plate assembly which has been tested many times, using a wide variety of different techniques, and has been modeled with various finite element codes. As a result, its modal characteristics are well documented.

The model was tested by mounting a reference transducer on one of the upper corners of the vertical member (point number 53). A roving transducer was used to measure the response at a grid of points distributed over the model. In Figure 5-25, a schematic drawing of the T-plate structure is shown with the testing points marked. An impact force was applied at 25 points randomly selected over the structure.

An impact force, provided it supplies energy in the frequency range of interest, will normally satisfy the input assumptions as stated in Section 2.3. In order to demonstrate this condition for the T-plate example, a typical force spectrum was measured and is shown in Figure 5-26. As can be seen from this figure, the force spectrum is nearly uniform in the frequency range of interest.

In Figure 5-27, a complex exponential curvefit of a typical response signal is shown. The complex exponential so closely approximates the response signal that it is not possible to distinguish the difference in the plot. This figure demonstrates the fact that existing frequency response function curve fitting algorithms can be used to approximate response function data.
Figure 5-25: Reference Transducer Position - TPLATE
Figure 5-26: Typical Impact Force Spectrum
The poles and residues estimated by this curve fitting routine are system characteristics.

The actual residue estimation scheme used in this T-plate example was not nearly as sophisticated as the complex exponential method. The technique used was the Hann-amplitude method because it is directly compatible with existing Fourier analyzers. The residue is estimated from the amplitude at the resonance frequency of the Fourier spectrum of the response data after a Hann window has been applied to the history.

The main advantage of the Hann-amplitude method is its simplicity of computation for both the modal scale factor and modal assurance criterion from the response ratio function. Using the Hann-amplitude method, the modal scale factor and modal assurance criterion are calculated by the frequency response and coherence function algorithms that are standardly implemented in a Fourier analyzer. It should be noted that the Hann-amplitude method is sensitive to measurement noise, particularly for heavily damped systems, but works very well for lightly damped systems like the T-plate.

The Hann-amplitude method was used only for the residue estimates. The least squares version of the complex exponential was used to determine the system poles. The system poles were determined by impacting the structure at the twenty-five points, computing the averaged autocorrelation of the response histories,
Figure 5-27: Complex Exponential Curvefit of Response Spectrum Data
and executing a curvefit with the complex exponential algorithm.

The modal scale factor was determined by setting up the Fourier analyzer to perform a standard frequency response calculation of the roving-point response. The analyzer was triggered from the reference signal and a Hann window was applied to both signals. The frequency response and coherence were computed for these two signals for the twenty-five randomly applied impacts.

A typical measurement is shown in Figure 5-28. The auto moment of the reference response is shown at the bottom, the modal assurance criterion is shown in the middle, and the response ratio function at the top. The peaks in the auto spectrum are located at the frequencies of the system poles. As can be seen from this figure, the modal assurance criterion is equal to unity at these frequencies. The value of the modal scale factor can be determined directly from the response ratio function since it corresponds to the modal scale factor computation.

It is particularly interesting to note the flat characteristic of the modal assurance criterion function and the modal scale factor function in Figure 5-28. This characteristic of these functions demonstrates that, in the absence of closely spaced modes, the value of the response ratio function does not change in the neighborhood of the system pole. This information correlates with the observation that many single degree of freedom techniques are insensitive to the estimate of the system pole in
the determination of the modal vector.

The power of this technique is that the modal properties are measured directly from a large number of force input conditions. Where the modal assurance criterion value is unity, the system modal scale factors are unique and correspond to the system modal vectors at the measured points.

In Figure 5-29, the modal vectors are plotted for two closely spaced poles of the T-plate. These modes are somewhat difficult to separate using single degree of freedom methods. For this case the Hann-amplitude technique with the modal scale factor method worked very well. This is indicated by the modal assurance criterion value of unity for both modes, and by comparing these modes with those of previous test and finite element results.

5.6.5.2 REPEATED POLE EXAMPLE

The second example indicates another attractive feature of the modal assurance criterion approach: the ability to detect repeated system poles. The test article was a metal circular cylinder, with annular cross-section, symmetric about its axial direction (in the shape of a stubby section of pipe). Repeated roots could be expected for modes involving radial motion, as a consequence of this symmetry.
Figure 5-28: Typical Response Ratio Measurements - TPLATE
Figure 5-29: Typical TPLATE Modal Vectors
The same test procedure was used as for the T-plate. A reference transducer position was chosen and a roving transducer was used to measure the response around a plane perpendicular to the axis of the cylinder. The force input consisted again of twenty-five randomly distributed impacts over the surface of the cylinder.

In Figure 5-30, a plot of the response auto spectrum and modal assurance criterion are shown for a typical measurement. In this case, except for one small interval of frequency, the value of modal assurance criterion is considerably less than unity, even though the system poles seem to be very widely separated in frequency. This means that the values of the modal scale factor measured for the modes where the modal assurance criterion is less than unity are not unique. This could be caused by a repeated root situation or by measurement noise. Since the measurement technique was equivalent to that used for the T-plate and proper care was taken, the repeated root case is likely.

Figure 5-31 displays a zoom Fourier transform performed around one of the resonances suspected to be a repeated root. The zoom bandwidth was 25 Hertz, centered around a frequency of approximately 3200 Hertz. There is no visual evidence in the figure that two poles exist in this frequency band. In Figure 5-32, a plot of the auto moment and the modal assurance criterion for a zoom Fourier transform range of 400 Hertz are shown.

Two modal vectors were measured at this frequency using a
Figure 5-30: Typical Response Ratio Measurements - RING
Figure 5-31: Typical Response Ratio Measurements - RING
Increased Frequency Resolution
conventional frequency response method, one for each of two different exciter positions. Both are one-diameter ring modes of the cylinder that have an antinode at the position of the excitation force. This is a classic characteristic of a repeated root. In other words, when there is no unique mode shape for a system pole, a linear combination of two mode shapes is measured.

Two modal vectors for the pole which had the modal assurance criterion value equal to unity were also measured, using conventional procedures with two exciter positions. The modal vectors are identical, within experimental error, and illustrate a "breathing" mode shape of the cylinder. A value of unity for the modal assurance criterion indicates that such uniqueness should occur.

In order to demonstrate more clearly the measurement characteristics of a repeated root, a small mass was attached to the cylinder to destroy its cylindric symmetry, and split the repeated root into two separate poles. The MAC was measured for this configuration and the results are shown in Figure 5-33. The computed MAC value is now near unity for most of the modes of the cylinder, but is still less than unity for the one-diameter mode. In order to explain this result, a zoom Fourier transform was again processed as before, and two poles are clearly evident in the data. The addition of mass results in a frequency separation of the poles of only 14 Hertz as shown by Figure 5-34. The
measurement process that produced the modal assurance criterion plot of Figure 5-33 had a frequency resolution too coarse to resolve the separation of the poles. Therefore, a result was computed that indicated a repeated root. In other words, the poles are distinct, but are apparently identical under coarse observation. In fact, it seems likely that true, exactly repeated system poles seldom exist in physical structures, but that poles can often be so nearly identical that they appear to be repeated system poles in the measurements, and cannot be separated by common modal parameter estimation algorithms.

The modal assurance criterion function was recalculated using the same frequency resolution as in the zoom measurement with the results shown in Figure 5-35. As can be seen, the modal assurance criterion value is equal to unity for each of the two modes. The modal vectors were then measured with this finer frequency resolution. The two modal vectors are one-diameter ring modes with orientations (nodal lines) rotated 90 degrees with respect to each other as might be expected.
Figure 5-34: Possible Repeated System Pole - RING+MASS
Increased Frequency Resolution
Figure 5-35: Typical Response Ratio Measurements - RING+MASS
Increased Frequency Resolution
REFERENCES


5. Guyan, R.J. "Reduction of Stiffness and Mass Matrices" AIAA Journal, Volume 3, Number 2 February 1965, pp. 380


6.1 OVERVIEW

The multiple input estimation of frequency response functions is desirable as a potential experimental modal analysis technique for several reasons. Obviously, the concepts of modal scale factor and modal assurance criterion, developed in the last chapter, were formulated to provide methodology for evaluating modal vectors estimated from different rows or columns of the frequency response function matrix. While the concepts of modal scale factor and modal assurance criterion can take advantage of multiple input estimation of frequency response functions, they do not require simultaneous multiple inputs.

The principal advantage expected for the use of multiple input estimation of frequency response functions is the increase in accuracy of the frequency response function estimates. During single input excitation of a system, there may exist large differences in the amplitude of vibratory motion at various locations because of the dissipation of the excitation power
within the structure, especially in the case of heavy damping. Small nonlinearities in the structure will consequently cause errors in the measurement of the response. Often this type of error is exhibited as an overestimation of the structural compliance in the vicinity of the excitation location, due to a softening nonlinearity at that point. With multiple input excitation, the vibratory amplitudes across the structure will typically be more uniform, with a consequent decrease in the effect of nonlinearities.

A second reason for the expectation of improved accuracy in the measurement of frequency response functions is that the estimates generated in the case of multiple input excitation are more likely to be consistent with each other than for the single input case. When a number of exciter systems are used, elements from columns of the frequency response function matrix, corresponding to those exciter locations, are being determined simultaneously. When each column is determined independently, it is possible for small errors of measurement due to nonstationary inputs and time-dependent system characteristics to cause a shift in resonance frequencies and a distortion of modal vectors of the system.

Another advantage of the simultaneous measurement of a number of columns of the frequency response function matrix is the ability to use a linear combination of frequency response functions in
the same row of the matrix in order to enhance specific modes of the system. This technique is analogous to the forced normal mode excitation experimental modal analysis in which a structure is excited by a forcing vector which is proportional to the modal vector of interest. For this analysis, the coefficients of a preliminary experimental modal analysis are used to weight the frequency response functions, so that the sum emphasizes the modal vector that is sought. The revised set of conditioned frequency response functions is analyzed to improve the accuracy of the modal vector. The common example of this approach for a system with approximate structural symmetry would be to excite at two symmetric locations. The sum of the two frequency response functions at a specific response location should enhance the symmetric modal vectors. Likewise, the difference of the two modal vectors should enhance the antisymmetric modal vectors.

6.2 MULTIPLE INPUT THEORY

The theoretical basis of multiple input/output frequency response function analysis is well documented in a number of sources [1-6]. Some of the more recent work in the development of the theory has been concerned with a concise matrix formulation of ordinary, partial, and multiple coherence [3,4,5]. While much
has been written about multiple input/output theory, very little experimental work has been documented. The experimental work that has been documented is primarily concerned with the acoustic area. The application of the multiple input/output to the experimental modal analysis area has apparently not been seriously investigated. Prior to the availability of fast Fourier analysis equipment, the multiple input estimation approach was not very practical.

Consider the case of n inputs and m outputs measured during a modal test on a dynamic system as shown in Figure 6-1. The model assumed for the dynamics is:

\[
Y(i) = \sum_{j=1}^{n} H(i,j) X(j) + N(i)
\]

where:

- \(Y(i)\) = Spectrum of the i-th output
- \(X(j)\) = Spectrum of the j-th input
- \(H(i,j)\) = Frequency response function of output i with respect to input j
- \(N(i)\) = Spectrum of the noise
  - Part of \(Y(i)\) not linearly related to \(X(j)\)

The least squares method for determining the frequency response functions is derived by minimizing the magnitudes of the noise spectra. The solution of this minimization process provides the matrix equation:
Figure 6-1: Multiple Input System Model
\[
(6-2) \quad [GYX] = [H] [GXX] + [Z]
\]

where:
- \([GYX]\) = \(m \times n\) frequency dependent matrix
- \([GXX]\) = Input-output cross spectrum matrix
- \([GYX(i,j)]\) = Cross spectrum of output \(i\) with respect to input \(j\)
- \([GXX(i,j)]\) = Cross spectrum of input \(i\) with respect to input \(j\)
- \([H]\) = \(m \times n\) frequency response function matrix
- \([Z]\) = \(m \times n\) frequency dependent matrix
- \([Z]\) = Noise cross spectrum matrix.

In the experimental procedure, the input and response signals are measured, and the averaged cross spectra and auto spectra necessary to create the \([GYX]\) and \([GXX]\) matrices are computed. If the computation of ordinary, multiple, or partial coherence functions will be required, then the output cross spectrum matrix \([GYY]\) must be formulated.

The least squares estimate of the frequency response matrix is computed as if the noise cross spectrum matrix were zero, and so the formal solution yields:
(6-3) \[ [H] = [GYX] [GXX]^{-1} \]

Note that the input cross spectrum matrix must be inverted at every frequency in the analysis range. This means that the "computational load" on the analysis system is much greater than for the single input case, in which only the reciprocal of a single input autospectrum is computed.

Equation 6-3 is valid regardless of whether the various inputs are or are not correlated. Unfortunately, there are a number of situations where the input cross spectrum matrix \([GXX]\) may be singular at specific frequencies or frequency intervals. When this happens, the inverse of \([GXX]\) will not exist and Equation 6-3 cannot be used to solve for the frequency response function at those frequencies or in those frequency intervals. First, one of the input autospectra may be zero in amplitude over some frequency interval. When this occurs, then all of the cross spectra in the same row and column in the input cross spectrum matrix \([GXX]\) will also be zero over that frequency interval. Consequently the input cross spectrum matrix \([GXX]\) will be singular over that frequency interval.

Second, two or more of the input signals may be fully coherent over some frequency interval. This can be detected by computing the ordinary coherence function between the inputs. The inputs are fully coherent when their ordinary coherence function is
equal to unity. For example, assume that the i-th and j-th inputs are fully coherent. If $G(i,j)$ is defined to be the input cross spectrum of $X(i)$ with respect to $X(j)$, $GXX(i,j)$, divided by the input auto spectrum of $X(i)$, $GXX(i,i)$, then when the ordinary coherence is unity, the result is as follows:

\[(6-4) \quad X(i) = G(i,j) \quad X(j)\]

\[(6-5) \quad GXX(j,k) = G(i,j) \quad GXX(i,k)\]

\[(6-6) \quad GXX(k,j) = G^*(i,j) \quad GXX(k,i)\]

Therefore, the input cross spectrum matrix $[GXX]$ is singular.

Last, numerical problems, which cause the computation of the inverse to be inexact, may be present. This can happen when an autospectrum is near zero in amplitude, when two or more inputs are highly coherent (their ordinary coherence is near unity), when the cross spectra have large dynamic range with respect to the word size used to store the functions in the computer, or when the matrix is ill-conditioned because of nearly redundant input signals.

The concept of coherence function, as defined for the single input case, needs to be expanded to include the variety of
relationships that are possible in the multiple input case. Ordinary coherence is defined as the correlation coefficient describing the possible causal relationship between any output and any input, oblivious of all other outputs and inputs. Under this definition there will be an ordinary coherence function for every input/output combination. This is exactly the coherence function that is defined for the single input situation. Great care must be taken in the interpretation of ordinary coherence when more than one input is present.

Partial coherence is defined as the ordinary coherence between a conditioned output and a conditioned input. The output and input are conditioned by removing the potential contributions to the output and input from other input(s). The removal of the effects of the other input(s) is formulated on a linear least squares basis where the order of removal has a definite effect upon the partial coherence if some of the input(s) are mutually correlated. There will be a partial coherence function for every input/output combination for all permutations of conditioning. The primary value of partial coherence, then, is to ascertain whether some unknown input, correlated with a known input, is contributing to the output. The usefulness of partial coherence with respect to experimental modal analysis appears to be of limited value since all inputs are, ideally, known and since the degree of mutual correlation can be determined from the ordinary coherence computed between any combination of inputs. For these
reasons, partial coherence was not included as part of this investigation.

Multiple coherence is defined as the correlation coefficient describing the possible causal relationship between an output and all known inputs. There will be a multiple coherence function for every output. Therefore, multiple coherence can be used to evaluate importance of unknown contributions to each output. These unknown contributions can be measurement noise, nonlinearities, or unknown inputs. Particularly, as in the evaluation of ordinary coherence, a low value of multiple coherence near a resonance will often mean that the leakage error is present in the frequency response function. Unlike the ordinary coherence function, a low value of multiple coherence is not expected at the antiresonance. The multiple coherence is not sensitive to the leakage error in the region of the antiresonances since, from different inputs, the antiresonances do not occur at the same frequency. Thus, any response resulting from many inputs will consist of data which is more significant than the leakage contribution from adjacent resonances.

The ordinary coherence function can be formulated in terms of the elements of the matrices defined previously. The ordinary coherence function between the i-th output and the j-th input can be computed from the following formula:
\[ (6-7) \quad COH(i,j) = \frac{\left| GYX(i,j) \right|^2}{GXX(j,j) \cdot GYY(i,i)} \]

where: \( COH(i,j) \) = Ordinary coherence function between signal i and signal j

\([GYY]\) = \( m \times m \) matrix function of frequency

\([GYY]\) = Output cross spectrum matrix

\( GYY(i,j) \) = Cross spectrum between output i with respect to output j

In order to formulate the multiple coherence functions in a concise form, the equations can be greatly simplified by the formulation of a matrix created from the input cross spectrum matrix \([GXX]\) and the output spectrum matrix \([GYY]\). This matrix can be defined as the augmented input cross spectrum matrix for the i-th output \([GYXX(i)]\) as follows:

\[
(6-8) \quad [GYXX(i)] = \begin{bmatrix}
GYX(1,1) & GYX(1,1) & GYX(1,2) & GYX(1,3) \\
GYX(1,1) & GXX(1,1) & GXX(1,2) & GXX(1,3) \\
GYX(2,1) & GXX(2,1) & GXX(2,2) & GXX(2,3) \\
GYX(3,1) & GXX(3,1) & GXX(3,2) & GXX(3,3)
\end{bmatrix}
\]

Therefore, as a check on the determination of the frequency
response matrix, the multiple coherence functions can be computed from the following formula:

\[
MCOH(i) = 1 - \frac{\text{DET}(GYY(i,i))}{\text{DET}(GXX)}
\]

(6-9)

If the multiple coherence of the i-th output is near unity, then the i-th output is well predicted from the set of inputs using the least squares frequency response functions.

Of the variety of situations that can cause difficulties in the computation of the frequency response functions, the most troublesome is the case of coherent inputs. If two of the inputs are fully coherent, then there are no unique frequency response functions associated with those inputs. This can be measured by computing the ordinary coherence function between each pair of inputs. Contrary to the frequency response case, it is desirable that the ordinary coherence function should not be unity anywhere within the frequency range of interest. Although the signals used as input to the exciter systems can be uncorrelated random inputs, the response of the structure at resonance combined with the inability to completely isolate the exciter systems from this response will result in an ordinary coherence function with values other than zero, particularly at the system poles. As long as the ordinary coherence function is not unity at these frequencies, Equation 6-4 will give the expected result.
6.2.1 DUAL INPUT CASE

For the purpose of investigating the potential of the multiple input estimation approach to frequency response function formulation, the natural place to begin is with two inputs. Additionally, though, to go beyond the dual input situation would begin to approach the limitations of the equipment available at the current time at the University of Cincinnati. At this time, it is much more important to evaluate the general concept rather than to investigate all of the possible alternatives. Based upon the experimental results, a number of future areas of research will be delineated. Rather than work with the matrix representation of the previous section, the equations will become more readily understood for the case of two inputs if the matrices can be reduced appropriately. Therefore, the following equations represent a restatement of the relationships of the previous section for the \( i \)-th output. The basic equations representing the model used to describe the dual input case as shown in Figure 6-2 is found from Equation 6-2.

\[(6-10) \quad Y(i) = H(i,1) \ X(1) + H(i,2) \ X(2) + N(i)\]

The least squares estimates of the frequency response functions can be computed as if the noise cross spectrum matrix is zero.
Figure 6-2: Dual input System Model

This results in the following matrix inversion equation.

\[
\begin{bmatrix}
H(1,1) & H(1,2) \\
H(2,1) & H(2,2) \\
H(3,1) & H(3,2) \\
\cdots & \cdots \\
H(m,1) & H(m,2)
\end{bmatrix}
\begin{bmatrix}
GYX(1,1) & GYX(1,2) \\
GYX(2,1) & GYX(2,2) \\
GYX(3,1) & GYX(3,2) \\
\cdots & \cdots \\
GYX(m,1) & GYX(m,2)
\end{bmatrix}
= \begin{bmatrix}
GYX(1,1) & GYX(1,2) \\
GYX(2,1) & GYX(2,2)
\end{bmatrix}^{-1}
\begin{bmatrix}
GXX(1,1) & GXX(1,2) \\
GXX(2,1) & GXX(2,2)
\end{bmatrix}
\]
Equation 6-11 can now be simplified by computing the matrix inverse resulting in the following two equations:

\[
(6-12) \quad H(i,1) = \frac{GYX(i,1) GXX(2,2) - GYX(i,2) GXX(2,1)}{\text{DET}[GXX]}
\]

\[
(6-13) \quad H(i,2) = \frac{GYX(i,2) GXX(1,1) - GYX(i,1) GXX(1,2)}{\text{DET}[GXX]}
\]

where: \( \text{DET}[GXX] = \) Determinant of \([GXX]\) matrix

\[
\text{DET}[GXX] = GXX(1,1) GXX(2,2) - GXX(2,1) GXX(1,2)
\]

Three equations for ordinary coherence can be formulated for the dual input situation. While the interpretation of these ordinary coherence functions is not always straightforward, the computation is relatively trivial.

\[
(6-14) \quad \text{COH}(i,1) = \frac{|GYX(i,1)|^2}{GXX(1,1) GYY(i,i)}
\]

\[
(6-15) \quad \text{COH}(i,2) = \frac{|GYX(i,2)|^2}{GXX(2,2) GYY(i,i)}
\]

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Equations 6-14 and 6-15 represent the ordinary coherence functions between the i-th output and the two inputs. Equation 6-16 represents the formulation of the ordinary coherence function between the two inputs. The multiple coherence function for the i-th output formulated from Equation 6-9 will require the augmented input cross spectrum matrix \([GYXX]\) of Equation 6-8. Restating Equation 6-8 for the dual input case gives:

\[
(6-17) \quad [GYXX(i)] = \begin{bmatrix}
GYY(i,i) & GYX(i,1) & GYX(i,2) \\
GYX(1,i) & GXX(1,1) & GXX(1,2) \\
GYX(2,i) & GXX(2,1) & GXX(2,2)
\end{bmatrix}
\]

Finally, the multiple coherence function can be formulated from Equation 6-9 as follows:

\[
(6-18) \quad MCOH(i) = \frac{\text{NUM}}{\text{DEN}}
\]
where: $\text{NUM} = GXY(i,1) \cdot GXY(1,i) \cdot GXX(2,2)$

- $GXY(i,1) \cdot GXY(2,i) \cdot GXX(1,2)$

+ $GXY(i,2) \cdot GXY(2,i) \cdot GXX(1,1)$

- $GXY(i,2) \cdot GXY(1,i) \cdot GXX(2,1)$

$\text{DEN} = GYY(i,i) \cdot GXX(1,1) \cdot GXX(2,2)$

- $GYY(i,i) \cdot GXX(2,1) \cdot GXX(1,2)$

Simplifying this equation yields the final result:

\[
(6-19) \quad \text{MCOH}(i) = \frac{\begin{vmatrix} GXY(i,1) & GXX(2,2) \\ + & GXX(1,1) \\ - GXY(i,1) \cdot GXX(2,i) \cdot GXX(1,2) \\ - GXY(i,2) \cdot GXX(1,i) \cdot GXX(2,1) \end{vmatrix}}{\begin{vmatrix} GYY(i,i) \cdot GXX(1,1) \cdot GXX(2,2) \\ - GYY(i,i) \cdot GXX(2,1) \cdot GXX(1,2) \end{vmatrix}}
\]

6.3 EXPERIMENTAL APPROACH

The estimation of the frequency response functions for the dual input case can proceed in one of at least two ways. Obviously from Equation 6-4, if the terms off the diagonal in the input
cross spectrum matrix are zero, the solution is trivial. This approach can be represented by the concept that the excitation must be totally uncorrelated. Secondly, if the off diagonal terms of the input cross spectrum matrix are not zero, the matrix can be inverted and the frequency response functions can be computed from Equations 6-12 and 6-13.

6.3.1 UNCORRELATED EXCITATION

There are at least three approaches to achieving uncorrelated excitation for the estimation of frequency response functions. The first technique for uncorrelated excitation is the use of uncorrelated random signals for the n excitation functions. In this case a greater averaging period is required for the measurement process, because for each frequency response measurement, n-1 of the excitation signals will appear as high-amplitude measurement noise. A speed increase is possible with this technique if the averaging time required is less than n times the normal measurement period. It is particularly important to note, though, that the theoretical uncorrelated input signals are uncorrelated only as the number of averages approaches infinity. For any finite number of averages, there will be some degree of correlation even in the absence of impedance mismatch or other complicating factors.
The second technique for uncorrelated excitation occurs when the two excitation inputs, $X(i)$ and $X(k)$, occupy nonoverlapping regions of the frequency range used for the analysis. Their cross spectrum will be zero and, hence, the signals will be uncorrelated. This method of coverage of the total bandwidth can be performed in a variety of ways. One possible technique is the division of the total range into $n$ equal bands, and another is the interweaving of a number of very narrow bands such that every $n$-th band is associated with the same excitation signal. In either case, this method will require the use of $n$ measurement periods. For each period the ordering of the bands is rearranged (for example, in a cyclic fashion) so that after $n$ periods each excitation signal has covered the entire frequency range. If the range is divided into $n$ equal bands, then there are $n-1$ regions in the frequency range in which measurement errors can be introduced because of the discontinuity in the excitation spectrum, but the information between these regions will be adequately determined.

No speed advantage can be expected from these two uncorrelated excitation techniques since $n$ full measurement periods will be required. For the interwoven excitation method, the number of regions of discontinuity will be large, but because of the unavoidable presence of some nonlinearity in the exciter systems and the structure being tested, the magnitude of the discontinuity may not be severe. Additionally, the "leakage" of
excitation energy into the frequency bands associated with another excitation signal may allow the use of a shorter measurement period in each of the n phases, so that a slight speed advantage may be obtained.

The third technique involves the instrumental variable approach outlined in Section 3.4. This impedance isolation approach utilizes the desired input spectrum to reduce the impedance mismatch by way of digital signal processing. While this method is very attractive, computational problems exist if the impedance mismatch is great. The computational problems can arise due to the limited word size of the minicomputers used in the digital signal processing. In situations where this impedance isolation approach can be used, a speed advantage over the other uncorrelated excitation methods can be expected.

In the scope of this research, only the first approach to uncorrelated excitation was attempted. The results of this approach were disappointing, both in terms of the ability of the multiple input estimates to approach the single input estimates of the frequency response functions and in the time required to obtain even these limited results.
6.3.2 MATRIX INVERSION

The experimental approach based upon the solution of Equation 6-3 only requires that the inputs not be perfectly correlated at any frequency. In order to try to achieve this condition, the approach to uncorrelated excitation using totally incoherent random signals for each of the inputs can be used. Even though there will be some degree of correlation due to the impedance mismatch between the excitation systems and the structure and also due to the finite averaging time, as long as the ordinary coherence function between the inputs is not unity at any frequency of interest, the frequency response functions can be formulated as in Equation 6-12 and Equation 6-13. Additionally, the value of the determinant of the input cross spectrum matrix \([GXX]\) can be used as check for singularity and, therefore, existence of the inverse of the input cross spectrum matrix.

6.4 DUAL INPUT APPLICATIONS

The results of the dual input estimation of frequency response functions are comparable to the results of current single input
estimation of frequency response functions. The examples in the following sections show that the differences in the frequency response estimates are essentially indistinguishable. The multiple coherence functions exhibit characteristics that indicate, for lightly damped structures, leakage continues to be a problem. The leakage error, though, appears to be no more severe than in the single input case. The most significant result is that, for the structures tested, no significant increase in total time per measurement cycle was required to obtain frequency response functions that appear to be equivalent. The dual input estimation approach is obviously extremely attractive, then, since two frequency response functions are formulated per measurement cycle. This has the potential for greatly reducing the total test time when a large number of columns of the frequency response function matrix are desired. Additionally, the consistency of the frequency response functions within rows of the frequency response function matrix allows the enhancement of modes through the weighted summation of frequency response functions within rows of the frequency response function matrix.

6.4.1 UNCORRELATED EXCITATION

The use of uncorrelated excitation as a sole approach to the
estimation of frequency response functions for the case of two inputs was investigated on an automotive frame represented in Figure 6-3. This frame was chosen as a representative case since the modes are very lightly damped and the system is less flexible than many potential test structures anticipated in the aerospace area. The damping and flexibility are important in the evaluation of the impedance matching between the excitation system and the test structure.

The concept of uncorrelated excitation involves uncorrelated random signals as inputs to each of the two excitation systems. The intent of this approach is to eliminate the off diagonal terms in the input cross spectrum matrix by averaging enough time histories so that these terms are negligible. This will be the situation if the excitation systems are impedance matched to the test structure. If a sufficient impedance mismatch occurs, these off diagonal terms will remain significant. The impedance matching between the excitation systems and the test structure can be affected by the feedback characteristics of the excitation system. If voltage feedback operation is utilized in the excitation systems, the random input spectrums will govern the velocity characteristics generated by the exciters. If current feedback operation is utilized in the excitation systems, the random input spectrums will govern the force characteristics generated by the exciters. For this reason, both the case of voltage and current feedback need to be investigated.
Figure 6-4 shows three frequency response functions for a response at point 13 due to a excitation at point 1. For these plots the excitation system(s) were configured in voltage feedback operation. The plot in Figure 6-4a represents the single input situation. The plot in Figure 6-4b represents the single input situation with both excitation systems attached to the test structure but with only one excitation system operating. The plot in Figure 6-4c represents the dual input case for uncorrelated inputs. Even in the presence of many more averages, the estimate of the frequency response function shown in Figure 6-4c is not acceptable.

Figure 6-5 shows exactly the same information as Figure 6-4 with the exception that the excitation system(s) were configured in current feedback operation. This comparison shows nearly the same results as the data in Figure 6-4. Frequency response functions can be chosen for other input/output or other input/input combinations which do not show as severe of a problem but the test procedure can not identify these combinations prior to the test. For this reason, uncorrelated excitation as a complete answer to the dual input estimation of frequency response functions is not practical.

In order to ascertain whether the dual input estimation of frequency response functions will be practical in any form, the determination of the ordinary coherence function between the
Figure 6-4: Typical Frequency Response Function - FRAME Voltage Feedback Operation
Figure 6-5: Typical Frequency Response Function - FRAME Current Feedback Operation
inputs can provide further information. If the ordinary coherence function between the inputs indicates that the inputs are totally correlated at any specific frequency, there can be no unique determination of the frequency response functions. For this to be true, the ordinary coherence function must be unity. Figure 6-6 represents the ordinary coherence functions for the dual input estimates of the frequency response functions.

Figure 6-6a is the ordinary coherence function for the voltage feedback operation. Figure 6-6b is the ordinary coherence function for the current feedback operation. In both cases, although the ordinary coherence function approaches unity at certain frequencies, the ordinary coherence function between the inputs does not reach unity. This indicates that the matrix inversion procedure may offer some improvement in the estimation of the frequency response functions for the dual input case.

6.4.2 MATRIX INVERSION

The use of the matrix inversion procedure for the estimation of frequency response functions for the case of dual inputs was investigated for the case of the automotive frame represented in Figure 6-3 and for the case of the Schweitzer sailplane represented in Figure 6-13. In both examples, the procedure was to use uncorrelated excitation with the same excitation systems.
Figure 6-6: Ordinary Coherence Function Computed Between Inputs
utilized in the previous example. Since little difference in the frequency response function estimates was noticed between voltage and current feedback operation, this variable was not investigated further. In addition, the matrix inversion procedure of Equation 6-11 was implemented to compute the frequency response function estimates.

Figure 6-7 through Figure 6-12 are frequency response functions for the automotive frame for the single input and dual input cases for selected input/output positions and are characteristic of the rest of the data. Figure 6-14 through Figure 6-19 are frequency response functions for the sailplane for the single input and dual input cases for selected input/output positions and are characteristic of the rest of the data. In both examples, the ordinary coherence function is plotted for the case of single inputs and the multiple coherence function is plotted for the case of dual inputs. It should be noted that the multiple coherence function appears to contain some irregularities (values greater than unity and less than zero). This is a function of the computation scheme used and is due to the block floating point arithmetic. Figure 6-20 through Figure 6-25 are plots of each of the functions required for the computation of the frequency response functions based upon the dual input case. The inputs for this case of the sailplane show very little coupling through the structure as was the situation for the automotive frame. This is reasonable since the sailplane
Single Input:
Input : 1
Output : 1

Figure 6-7: Frequency Response Function H(1,1) - FRAME
Multiple Input: 1-15
Input : 1
Output : 1

Figure 6-8: Frequency Response Function $H(1,1)$ - FRAME
Single Input:
Input : 13
Output : 1

Figure 6-9: Frequency Response Function $H(1,13) - \text{FRAME}$
Multiple Input: 1-13
Input : 13
Output : 1

Figure 6-10: Frequency Response Function H(1,13) - FRAME
Single Input:
Input: 1
Output: 23

Figure 6-11: Frequency Response Function $H(23,1) - FRAME$
Multiple Input: 1-13
Input  : 1
Output : 23

Figure 6-12: Frequency Response Function $H(23,1)$ - FRAME
Figure 6-13: SAILPLANE Representation
Single input
Input  : 54
Output : 1

Figure 6-14: Frequency Response Function $H(1,94) - SAILPLANE$
Multiple Input, No Compensation
Input : 94
Output : 1

Figure 6.15: Frequency Response Function $H(1,94) - SAILPLANE$
Multiple Input: 94-95
Input: 94
Output: 1

Figure 6-16: Frequency Response Function $H(1,94) - \text{SAILPLANE}$
Figure 6-17: Frequency Response Function $H(1,95) - SAILPLANE$
Multiple Input, No Compensation
Input : 95
Output : 1

Figure 6-18: Frequency Response Function H(1,95) - SAILPLANE
Multiple Input: 94-95
Input: 95
Output: 1

Figure 6-19: Frequency Response Function $H(1,95)$ - SAILPLANE
is more weakly coupled from wing to wing when compared to the automotive frame.

The results of the dual input estimation of frequency response functions are within the typical experimental variation expected between estimates determined by single input frequency response function methods. The multiple coherence functions indicate a problem with leakage near the system poles but the same difficulty is denoted in the ordinary coherence functions for the single input approach. It is particularly significant to note that the number of averages used in the dual input case is exactly the same as the number of averages used in the single input case. This is significant since two frequency response functions are generated per measurement cycle for the dual input case compared to one frequency response function for the single input case. The net result is a time saving of approximately two.

6.4.3 WEIGHTED SUMMATION

One distinct advantage of the dual input estimation of frequency response functions is that the pair of measurements that are determined simultaneously are consistent with one another. This means that these two frequency response functions can be involved in a weighted summation in order to enhance certain modes or
Figure 6-20: Typical Input/Input Auto Spectrum - SAILPLANE
Figure 6-21: Typical Input/Input Cross Spectrum - SAILPLANE
Figure 6-22: Typical Output Spectrum - SAILPLANE
Figure 6-23: Determinant of Input Cross Spectrum Matrix - SAILPLANE
Input-Output Cross Spectrum: GYX(1,94)

Figure 6-24: Typical Input/Output Cross Spectrum - SAILPLANE
Input-Output Cross Spectrum: GYX(1,95)

Figure 6-25: Typical Input/Output Cross Spectrum - SAILPLANE
eliminate other modes. For the case of the sailplane, the excitation points were chosen to be at symmetric locations on each of the wings. For all symmetric modal vectors, the modal coefficients of the excitation locations should be approximately equal for real modes of vibration. For all antisymmetric modal vectors, the modal coefficients of the excitation locations should be approximately equal but opposite for real modes of vibration. Therefore, if the two frequency response functions for the i-th response point relative to each of the input points are added, all symmetric modes will be enhanced and all antisymmetric modes will be eliminated. If the two frequency response functions for the i-th response point relative to each of the input points are differenced, the antisymmetric modes will be enhanced and the symmetric modes will be eliminated. If symmetric input locations were not available, weighting other than plus or minus unity would need to be employed. While this procedure can be utilized with the frequency response functions generated from two single input frequency response function tests at the same two excitation locations, the results will not be as satisfactory due to the inconsistency between the columns of the frequency response function matrix.

Figure 6-26 through Figure 6-33 represent typical frequency response functions that result from the summation and difference of the frequency response functions generated from the dual input procedure. The column of frequency response functions that
result from any weighted summation of the existing frequency response function columns are simply new columns of the frequency response functions matrix and can be used in any modal parameter estimation procedure for the determination of modal vectors. Figure 6-34 represents the symmetric modal vectors obtained by using the quadrature modal parameter estimation procedure on the frequency response function column generated as the summation of the two frequency response function columns found from the dual input estimation approach. Likewise, Figure 6-35 represents the antisymmetric modal vectors obtained by using the quadrature modal parameter estimation procedure on the frequency response function column generated as the difference between the two frequency response function columns found from the dual input estimation approach. It should be noted that in these two cases the use of single degree of freedom modal parameter estimation algorithms is simplified due to the reduction of the effective number of degrees of freedom in the resulting frequency response functions.
Multiple Input: 94-95  
Input : 94+95  
Output : 1

Figure 6-26: Frequency Response Function H(1,94+95) - SAILPLANE
Multiple Input: 94-95
Input      : 94-95
Output     : 1

Figure 6-27: Frequency Response Function H(1,94-95) - SAILPLANE
Multiple Input: 94-95
Input : 94+95
Output : 22

Figure 6-28: Frequency Response Function H(22,94+95) - SAILPLANE
Figure 6-29: Frequency Response Function $H(22, 94-95) - SAILPLANE$
Multiple input: 94-95
Input: 94+95
Output: 94

Figure 6-30: Frequency Response Function H(94,94+95) - SAILPLANE
Figure 6-31: Frequency Response Function \( H(94, 94-95) \) - SAILPLANE

Multiple Input: 94-95
Input : 94-95
Output : 94
Multiple Input: 94-95
Input : 94+95
Output : 95

Figure 6-32: Frequency Response Function H(95,94+95) - SAILPLANE
Figure 6-33: Frequency Response Function $H(95, 94-95)$ - SAILPLANE

Multiple Input: 94-95
Input : 94-95
Output : 95
Figure 6-34: Symmetric Modal Vectors - SAILPLANE
Figure 6-35: Antisymmetric Modal Vectors - SAILPLANE
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7.1 CONCLUSIONS

The frequency response function approach to experimental modal analysis has the potential of being the most general and the most accurate of the currently available experimental modal analysis methods. This potential can only be realized as a result of continued clarification of the current methodology associated with this approach. The results of this dissertation serve to further this course of action.

The initial work of this dissertation provides the background needed to evaluate current and future technology requirements related to the field of experimental modal analysis. The review of experimental modal analysis methods provides valuable information concerning what are, and what are not, advantages and disadvantages of the current methods. The summary of frequency response function estimation concepts, particularly related to the reduction of variance and bias errors, provides a similar basis for the development of this specific area.
As a result of this initial background work, the remainder of the dissertation provides a response to the need for continued refinement of the frequency response function experimental modal analysis technique. The concept of the enhanced frequency response function results in improved frequency/damping estimates due to a reduction in the effective number of degrees of freedom that must be considered by the modal parameter estimation algorithms. The concepts of modal scale factor and modal assurance criterion provide a rigorous approach to the evaluation of modal vectors generated from different testing conditions. The addition of the modal assurance criterion as a confidence factor associated with modal vector estimation is an extremely beneficial factor in the utilization of the redundant modal vector information contained in the frequency response function matrix.

Finally, the multiple input estimation of frequency response functions is an extremely attractive addition to the frequency response function experimental modal analysis technique. The ability to generate consistent frequency response function information would be important if that were the sole benefit of this approach. The consistency of the frequency response functions allow for the weighted summation of frequency response functions, within rows of the frequency response function matrix, in order to enhance particular modes of vibration producing similar results as the enhanced frequency response function
technique. This resultant reduction of the effective number of degrees of freedom again reduces the complexity of the modal parameter estimation problem. In addition, though, the fact that this consistent frequency response function information can be obtained with no significant increase in time required per measurement cycle is an equally important result. For the dual input case investigated compared to the single input case, this produces approximately twice the amount of frequency response function information in the same amount of total test time.

7.2 RECOMMENDATIONS

As a result of the work described by this dissertation, a number of general and specific recommendations regarding continuing research have been identified. Certainly, continued evaluation of all experimental modal analysis methods is necessary in order to provide hybrid development of experimental modal analysis methods as well as providing objectivity in the value analysis of new methods. Similarly, the need for further investigation of error reduction techniques related to frequency response function estimation can be justified, even though current techniques provide adequate results based upon current requirements. Specifically, the value and limitations of the concepts of
enhanced frequency response function, modal scale factor and modal assurance criterion, and multiple input estimation of frequency response functions can only be obtained after more experimental analysis. The initial work on these concepts has already shown great reward but further documentation of these concepts with respect to changes in damping, in modal density, in excitation signal, as well as other modal test parameters will identify the extent of future potential. The ultimate importance of these concepts will depend upon the gradual alteration of the ideas under scrutiny of real world test conditions.

Finally, all of the results of this dissertation indicate that there is a need for simultaneous multiple channel data acquisition equipment that is specially designed for experimental modal analysis. Particularly, with the increased ability to generate consistent columns of frequency response function information, through the multiple input estimation of frequency response functions, and the increased ability to handle the redundant modal vector information, through the use of modal assurance criterion, this type of hardware will find immediate application.
A number of averages, \( n \), of both input \( X \) and output \( Y \) at frequency increments within some band of interest are recorded. The input \( X \) and output \( Y \) are linearly related at each frequency and can be represented by the following linear model.

(A-1) \[ Y = H(y,x) X \]

The value of the frequency response function \( H(y,x) \) can be calculated at each frequency increment in such a way as to minimize the sum of the squared errors between the output \( Y \) and the product of the frequency response function \( H(y,x) \) and the input \( X \). For the following formulation, all notation will apply with regard to an arbitrary frequency \( f \). The notation for frequency will be dropped in favor of clarity. The squared error based upon Equation A-1 is:

(A-2) \[ E = \sum_{k=1}^{m} | Y(k) - H(y,x) X(k) |^2 \]

The absolute magnitude is used since these variables are generally complex quantities. For the following equation,
\[ (A-3) \quad E = \sum_{k=1}^{m} \begin{bmatrix} Y(k) - H(y,x) X(k) \\ Y^*(k) - H^*(y,x) X^*(k) \end{bmatrix} \]

where:
- \( X^* = \) Complex conjugate of \( X \)
- \( Y^* = \) Complex conjugate of \( Y \)
- \( H^* = \) Complex conjugate of \( H \)

the squared error term can be differentiated with respect to the frequency response \( H(y,x) \) or the complex conjugate of the frequency response \( H(y,x) \), and the resulting derivative set to zero. Thus, the normal equation is:

\[ (A-4) \quad \frac{\partial E}{\partial H^*(y,x)} = 0 = \sum_{k=1}^{m} \begin{bmatrix} Y(k) - H(y,x) X(k) \\ -X^*(k) \end{bmatrix} \]

Solving for \( H(y,x) \) gives:

\[ (A-5) \quad H(y,x) = \frac{\sum_{k=1}^{m} Y(k) X^*(k)}{\sum_{k=1}^{m} X(k) X^*(k)} \]

The cross spectrum between output and input \( G(y,x) \) and the auto spectrum of the input \( G(x,x) \) can be defined:
\( (A-6) \quad G(y,x) = \sum_{k=1}^{\infty} Y(k) X^*(k) \)

\( (A-7) \quad G(x,x) = \sum_{k=1}^{\infty} X(k) X^*(k) \)

Then

\( (A-8) \quad H(y,x) = \frac{G(y,x)}{G(x,x)} \)

In actual practice, the calculated values of the cross and auto spectrum are dependent upon the input signal utilized. Statistically, though, these separate estimates are not unique if the input is not stationary. Even so, with nonstationary or deterministic inputs, the estimate of frequency response is unique and, therefore, the least squares approach is valid.

If the system can be defined as in Figure 3-1, and allowing noise only to enter at \( N \), the common experimental situation of noise on the output can be summarized as:

\( (A-9) \quad Y = H(y,x) X + N \)

Therefore, using the least squares approach in the same manner as before:
(A-10) \[ \sum_{k=1}^{\infty} Y(k) X^*(k) = H(y,x) \sum_{k=1}^{\infty} X(k) X^*(k) \]
\[ + \sum_{k=1}^{\infty} N(k) X^*(k) \]

Restating Equation A-10 using the identities of Equations A-6 and A-7 gives:

(A-11) \[ G(y,x) = H(y,x) G(x,x) + G(n,x) \]

where: \[ G(n,x) = \sum_{k=1}^{\infty} N(k) X^*(k) \]
\[ G(n,x) = \text{Cross spectrum between noise and input} \]

Solving for the estimate of the frequency response function in the presence of noise on the output yields:

(A-12) \[ H(y,x) = \frac{G(y,x)}{G(x,x)} - \frac{G(n,x)}{G(x,x)} \]

Therefore, the error in the estimate of the frequency response function is the ratio of \( G(n,x) \) over \( G(x,x) \). Assuming that the noise is unrelated to the input, this approaches zero as the number of averages becomes large.

In order to evaluate whether the estimated frequency response function represents a consistent relationship between the input and output signals, a correlation coefficient can be formulated between the predicted output signal and the measured output.
signal.

The auto spectrum of the predicted output based upon the least squares estimation of the frequency response function can be developed using the linear model of Equation A-1.

\[
G(y,y) = \sum_{k=1}^{\infty} \begin{bmatrix} H(y,x) X(k) \\ \bar{a}^*(y,x) X^*(k) \end{bmatrix}
\]

(A-13)

\[
G(y,y) = \sum_{k=1}^{\infty} \frac{H(y,x)}{G(x,x)}
\]

(A-14)

The auto spectrum of the measured output can be developed based upon the model in Equation A-9 which includes a noise signal on the output.

\[
G(y,y) = \sum_{k=1}^{\infty} \begin{bmatrix} H(y,x) X(k) + N(k) \\ \bar{H}^*(y,x) X^*(k) + N^*(k) \end{bmatrix}
\]

(A-15)

\[
G(y,y) = \sum_{k=1}^{\infty} \frac{H(y,x)}{G(x,x) + H(y,x) G(n,x) + H(y,x) G(x,n) + G(n,n)}
\]

(A-16)

where:

\[
G(n,n) = \sum_{k=1}^{\infty} N(k) N^*(k)
\]

\[
G(n,n) = \text{Auto spectrum of the noise signal}
\]
Assuming once again that the noise is uncorrelated with the input, the cross spectrum terms between the noise and the input will approach zero as the number of averages becomes very large.

(A-17) \[ G(y,y) = |H(y,x)|^2 G(x,x) + G(n,n) \]

It is obvious comparing Equation A-14 to Equation A-17 that the auto spectrum of the measured output includes a term that is correlated with the input and a term that is not correlated with the input. The part of the measured output auto spectrum that is correlated with the input can be defined by the following relationship.

(A-18) \[ COH(y,x) G(y,y) = |H(y,x)|^2 G(x,x) \]

Using this definition, the part of the measured output spectrum that is not correlated with the input can be represented as:

(A-19) \[ |1 - COH(y,x)| G(y,y) = G(n,n) \]

The correlation coefficient COH(y,x) is called the (ordinary) coherence function, and describes the separation of the output spectrum into parts that are correlated and uncorrelated with respect to the input.
If the cross spectrum between the noise and the input, \( G(n,x) \), is not everywhere zero, the coherence function can only be estimated. The estimate of the coherence function can be formulated from Equation A-18.

\[
(A-20) \quad \text{COH}(y,x) = \frac{|H(y,x)|^2 G(x,x)}{G(y,y)}
\]

Using Equation A-8 to simplify Equation A-20, the final result is:

\[
(A-21) \quad \text{COH}(y,x) = \frac{|H(y,x)|^2}{G(x,x) G(y,y)}
\]
Cyclic signal averaging has the appearance of a comb in the frequency domain where for this reason, this process may be referred to as a comb digital filter. The teeth of the comb are band pass frequency ranges. This form of signal averaging is very useful for filtering periodic components from a noisy signal since the teeth of the filter are positioned at harmonics of the frequency of the sampling reference signal. This is of particular importance in signature applications where it is desirable to extract signals connected with various rotating members. This same form of signal averaging is particularly useful for reducing leakage during frequency response measurements and also has been used extensively for evoked response measurements in biomedical studies.

A very common application of cyclic signal averaging is in the area of analysis of rotating structures. In such a signature analysis application, the peaks of the comb filter are positioned to match the fundamental and harmonic frequencies of a particular rotating shaft or component. This is particularly powerful, since in one measurement it is possible to filter all of the possible frequencies generated by the rotating member from a
given data signal. With a zoom Fourier transform type of approach, potentially only one shaft frequency at a time can be examined depending upon the zoom power necessary to extract the shaft frequencies from the surrounding noise.

In the application of cyclic signal averaging to frequency response function estimates, the corresponding fundamental and harmonic frequencies are now simply the frequency resolution, $\Delta f$, and integer multiples of $f$. In this case, the spectra between each $f$ is reduced with an associated reduction of the bias error called leakage.

The implementation of cyclic signal averaging proceeds in a manner easily applicable to most fast Fourier transform analyzers. The cyclic averaged inputs and outputs are normally computed by simply summing successive time records with respect to a timing signal that is coherent with the frequency of the signal of interest. It is possible to utilize the time shift property of the Fourier transform to handle the case with non-successive time records. If a time record is delayed by an amount $t$ from the start of the first history, the spectra can be compensated using

\[
(B-1) \quad F\{x(t + t_o)\} = X(f) \exp(-j 2\pi n \frac{t}{t_o})
\]

where: $F\{x(t)\} = X(f)$
Therefore, for the situation with no delays between successive samples, the correction is a unit magnitude with zero phase.

The frequency analysis corresponds to the intersection of the $\sin(X)/X$ curve and the frequency domain sampling function. It should be noted that it is possible to multiply the time domain signal by a time domain window and change the characteristics of the "comb" filter shape from a $\sin(X)/X$ function of the window function. For this case a weighted signal averaging procedure is used.

The signal averaging algorithm with a rectangular window (Boxcar window) is:

\begin{equation}
\bar{y}(t) = \frac{1}{n} \sum_{i=0}^{n-1} y_i(t)
\end{equation}

where: $y_i(t) =$ Time sample

- $n =$ Number of time samples
- $\bar{y}(t) =$ Signal averaged time sample.

For the case where $t$ is continuous, the complete Fourier series coefficients become:
(B-3) \[ C(k) = \frac{1}{T} \int_0^T y(t) \exp(-j \omega t) \, dt \]

Substituting Equation B-2 into Equation B-3 yields:

(B-4) \[ C(k) = \frac{1}{T} \int_0^T \frac{1}{n} \sum_{i=0}^{n-1} y_i(t) \exp(-j \omega t) \, dt \]

or

(B-5) \[ C(k) = \frac{1}{nT} \sum_{i=0}^{n-1} \int_0^T y_i(t) \exp(-j \omega t) \, dt \]

Since \(y(t)\) is a continuous function, the sum of integrals can be replaced with an integral evaluated from 0 to nT. Therefore:

(B-6) \[ C(k) = \frac{1}{nT} \int_0^T y(t) \exp(-j \omega t) \, dt \]

Equation B-6 represents a discrete Fourier transform with a total time of nT. This discrete Fourier transform has a frequency resolution, then, of 1/nT but the frequency spacing of the measurement will be much larger, 1/T, due to the cyclic signal averaging process. The signal/noise ratio is determined by the longer the total time record, the smaller the frequency resolution and the narrower the bandwidth of the filter elements. For noncontinuous time records the above discussion holds
provided a coherent phase relation exists between the time samples. This would be the case for system with a tachometer signal or some other type of coherent trigger signal.

LEAKAGE REDUCTION APPLICATION

Since the frequency resolution can be improved tremendously using this technique, while the frequency spacing is kept the same, it is apparent that the leakage errors can be reduced for a given measurement band. The comb filter in the frequency domain consists of $\sin(X)/X$ terms spaced at the frequency intervals determined by $1/T$. Depending upon the number of averages, the $\sin(X)/X$ terms are reduced in width in direct proportion to the total sample time $nT$. Figure B-1 is an example of this characteristic. In Figure B-2 the worst case frequency domain line shape is shown for the "Boxcar" window case (case where asynchronous signal averaging is used). As can be seen in Figure B-1, the worst case side bands of the "Boxcar" window are reduced by signal averaging of the time domain data. The window causes the leakage effects to drop sharply next to the measured frequency component. The curve plotted in Figure B-2 shows the skirts of the "Boxcar" window for different numbers of cyclic averages. The rapid drop near the frequency of the unknown sine components indicates that good values can be measured near the peaks but that the skirts will cover up some of the low amplitude data several spectral lines from the peak. For frequency
Figure B-1: Effective Filter Shape for Cyclic Averaging
Boxcar Weighting
Figure B-2: Filter Characteristics for Cyclic Averaging Boxcar Weighting
response data this could correspond to the antiresonances of a frequency response measurement.

In Figure B-3, the frequency domain line shape is shown for a Hann window being applied to the signal data before it is averaged. This averaging process is shown in Figure B-4. As can be seen in Figure B-3, the skirts drop off very rapidly. Therefore, for frequency response measurements Hann weighted signal averaging should drastically reduce the leakage errors which can exist when using broadband random excitation techniques to measure frequency response. In Figure B-5 through Figure B-8, a series of measurements were performed by measuring the frequency response of a very lightly damped automotive frame. The value of N indicates the number of cyclic time records averaged together and M is the number of asynchronous auto and cross spectrum averages: a total of NM time records were sampled. The first measurement was made by using a "Boxcar" window and normal processing to compute the frequency response (N=1, M=80). As can be seen for this case, the coherence function indicates there is considerable noise in the measurement. This noise could be coming from a number of potential sources, one of these sources being leakage. In Figure B-6, a Hann window was used to compute the frequency response (N=1, M=80). A significant improvement can be noted in the frequency response and coherence function measurement which indicates that there is significant leakage noise. In Figure B-7, a time averaged "Boxcar" window
Figure B-3: Filter Characteristics for Cyclic Averaging Hann Weighting

B-9
Figure B-4: Cyclic Averaging Application of Hann Weighting
function is used to compute the frequency response \((N=4, M=20)\). A very definite improvement can be noted when compared to the \(N=1\) case of the "Boxcar" data. By observing the coherence function at the frequency of the peaks in the frequency response, it is noticed that leakage effects are clearly reduced. In Figure B-8 a Hann window is used with the signal averaging \((N=4, M=20)\) and there is a tremendous reduction in the leakage. An interesting point is that the data near the antiresonance is also drastically improved due to the sharp roll off of the line shape of the Hann weighted averaging.
Figure B-5: Frequency Response Function
Figure B-6: Frequency Response Function with Hann Weighting
Figure B-7: Frequency Response Function with Cyclic Averaging
Figure B-8: Frequency Response Function with Hann Weighted Cyclic averaging
If Equation 5-7 is used as the basis for a model to calculate a least squares error estimate of the proportionality constant between rows or columns of the residue matrix, the model is linear as follows:

\[(C-1) \quad a(c,j,r) = \text{MSF}(c,d) \quad a(d,j,r)\]

In vector notation this would be:

\[(C-2) \quad \{A(c,r)\} = \text{MSF}(c,d) \quad \{A(d,r)\}\]

For the following formulation, all notation will be with regard to mode r. The notation for mode number will be dropped in favor of clarity but the development will apply to any arbitrary mode. Additionally, the evaluation of the modal scale factor and modal assurance criterion will proceed the same whether rows C and D are used or columns C and D. Thus, no distinction will be made as to column or row.

All of the elements of the modal vectors in rows/columns C and D exhibit the relationship stated in Equation C-2. The modal scale

C-1
factor is to be calculated so as to minimize the sum of the squared errors between corresponding elements of each modal vector. All or part of each modal vector can be used in such a calculation. Obviously, if some elements are consciously excluded a form of weighted least squares error estimation is involved. The sum of the squared error, $E$, is formulated as follows:

\[
(C-3) \quad E = \sum_{j=1}^{m} \left| C(j) - MSF(c,d) D(j) \right|^2
\]

where:
- $C(j) = \text{Modal vector from row/column C}$
- $D(j) = \text{Modal vector from row/column D}$
- $MSF(c,d) = \text{Modal scale factor between row/column C and row/column D}$
- $m = \text{Number of elements in each modal vector}$

Since Equation C-3 involves complex quantities, the evaluation of Equation C-3 must proceed using complex conjugates. Restating Equation C-3 in these terms gives:

\[
(C-4) \quad E = \sum_{j=1}^{m} \left[ C(j) - MSF(c,d) D(j) \right] \left[ \left[ C^*(j) - MSF^*(c,d) D^*(j) \right] \right]
\]
Equation C-4 can be differentiated with respect to MSF(c,d) or the complex conjugate of MSF(c,d) in the least squares development. Differentiating with respect to the complex conjugate of MSF(c,d) gives the following normal equation:

\[
\frac{\partial E}{\partial \text{MSF}^*(c,d)} = 0 = \sum_{j=1}^{m} \left[ C(j) - \text{MSF}(c,d) D(j) \left[ -D^*(j) \right] \right]
\]

Solving Equation C-5 for MSF yields:

\[
\text{MSF}(c,d) = \frac{\sum_{j=1}^{m} C(j) D^*(j)}{\sum_{j=1}^{m} D(j) D^*(j)}
\]

The numerator of Equation C-6 can be defined as the cross moment of the modal vectors. This will be represented by:

\[
\text{MOM}(c,d) = \sum_{j=1}^{m} C(j) D^*(j)
\]

Similarly, the denominator of Equation C-6 can be defined as the auto moment of the modal vectors. This will be represented by:

\[
\text{MOM}(d,d) = \sum_{j=1}^{m} D(j) D^*(j)
\]
Therefore, Equation C-6 can be restated in a more concise manner

\[(C-9) \quad \text{MSF}(c_r) = \frac{\text{MOM}(c,d)}{\text{MOM}(d,d)}\]

Equation C-9 implies that the modal vector of row/column D is a reference to which modal vector of row/column C is compared with. In the general case, modal vector C can be represented as the sum of two vectors. The first vector will be the portion of modal vector C that is linearly related to modal vector D. The second vector will be the remainder of modal vector C that is not linearly related to modal vector D and will be made up of contamination from other modal vectors, contamination from any random or bias errors in the frequency response function or modal parameter estimation. In other words, the relationship between modal vector C and modal vector D will have to be restated as:

\[(C-10) \quad \{C\} = \text{MSF}(c,d) \{D\} + \{N\}\]

Reformulating the least squares estimate of the modal scale factor, taking into consideration this error vector yields:

\[(C-11) \quad \text{MOM}(c,d) = \text{MSF}(c,d) \text{MOM}(d,d) + \text{MOM}(n,d)\]

where:

\[\text{MOM}(n,d) = \sum_{j=1}^{m} N(j) \cdot D(j)\]

\[\text{MOM}(n,d) = \text{Cross moment between the error}\]

C-4
vector and the reference modal vector

Solving Equation C-11 for the modal scale factor gives the result:

\[
(C-12) \quad \text{MSF}(c,d) = \frac{\text{MOM}(c,d)}{\text{MOM}(d,d)} - \frac{\text{MOM}(n,d)}{\text{MOM}(d,d)}
\]

In order to determine the importance of the error term, the likely source of contamination must be considered. In the process of estimating the modal vectors, errors in the measurement of the frequency response function as well as errors in the modal parameter estimation process will contribute to the overall error in the modal vector. These two sources of error will combine to present two probable forms of the error vector. First, the error vector can be random in distribution. If this is the situation, the magnitude of the contribution due to the error term should decrease as the number of elements used in the calculation increases. Second, the error vector can be biased in the form of a linear summation of other modal vectors of the system. Since each of these other modal vectors are theoretically orthogonal to the reference modal vector, this sort of bias should be uncorrelated with the reference modal vector. Again, as the number of elements involved in the calculation is increased, the contribution due to the error term should decrease.
The modal scale factor represents the best least squares linear complex valued constant which will reduce the error function to a minimum for a particular modal vector estimate with respect to a particular reference modal vector. The calculation of the modal scale factor does not imply that this complex valued constant represents a consistent linear relationship, position by position, throughout the estimated modal vector compared to the reference modal vector. For example, if the reference modal vector is a rigid body mode of vibration and the estimated modal vector is one of the deformation modes, a modal scale factor can be calculated but has no utilitarian value. In order to ascertain that the modal scale factor consistently represents the linear relationship between the reference modal vector and the estimated modal vector, a correlation coefficient must be developed that relates the estimated modal vector to a predicted modal vector. The predicted modal vector can be calculated from the reference modal vector and the complex valued modal scale factor. The following development of such a correlation coefficient closely parallels the development of the coherence function.

The auto moment of the predicted modal vector can be formulated from Equation C-2 as follows:
\( \text{(C-13)} \quad \text{MOM}(c,c) = \sum_{j=1}^{M} \begin{bmatrix} \text{MSF}(c,d) \, D(j) \\ \text{MSF}^*(c,d) \, D^*(j) \end{bmatrix} \)

\( \text{(C-14)} \quad \text{MOM}(c,c) = \text{MSF}(c,d) \, \text{MSF}^*(c,d) \, \text{MOM}(d,d) \)

Likewise, the auto moment of the estimated modal vector can be formulated from Equation C-10:

\( \text{(C-15)} \quad \text{MOM}(c,c) = \sum_{j=1}^{m} \begin{bmatrix} \text{MSF}(c,d) \, D(j) + N(j) \\ \text{MSF}^*(c,d) \, D^*(j) + N^*(j) \end{bmatrix} \)

\( \text{(C-16)} \quad \text{MOM}(c,c) = \text{MSF}(c,d) \, \text{MSF}^*(c,d) \, \text{MOM}(d,d) \\
+ \text{MSF}(c,d) \, \text{MOM}(d,n) \\
- \text{MSF}^*(c,d) \, \text{MOM}(n,d) \\
+ \text{MOM}(n,n) \)

Using relationships of complex conjugates in complex arithmetic:

\( \text{(C-17)} \quad \text{MSF}(c,d) \, \text{MSF}^*(c,d) = |\text{MSF}(c,d)|^2 \)

\( \text{(C-18)} \quad \text{MOM}(d,c) = \text{MOM}^*(c,d) \)
Substituting these identities into Equation C-14 and Equation C-16 will give:

(C-20) \( \text{MOM}(c,c) = |\text{MSF}(c,d)|^2 \text{MOM}(d,d) \)

(C-21) \( \text{MOM}(c,c) = |\text{MSF}(c,d)|^2 \text{MOM}(d,d) \)

\[ + 2 \text{MSF}(c,d) \text{MOM}(n,d) + \text{MOM}(n,n) \]

The cross moment between the error vector and the reference modal vector has already been shown to be insignificant in the presence of typical errors. If the noise is uncorrelated with the reference modal vector, the last two terms of Equation C-21 will become small in the presence of a large number of elements involved in the calculation. Therefore:

(C-22) \( \text{MOM}(c,c) = |\text{MSF}(c,d)|^2 \text{MOM}(d,d) + \text{MOM}(n,n) \)

Comparing Equation C-22 to Equation C-20 relates the separation of the auto moment of the estimated modal vector into portions correlated and uncorrelated with respect to the reference modal vector. If the modal assurance criterion is a scalar constant relating the portion of the auto moment of the modal vector
related to the reference modal vector, then the following equation is applicable:

\[(C-23) \quad MAC(c,d) \cdot MOM(c,c) = |MSF(c,d)|^2 \cdot MOM(d,d)\]

Therefore, solving for the modal assurance criterion:

\[(C-24) \quad MAC(c,d) = \frac{|MSF(c,d)|^2 \cdot MOM(d,d)}{MOM(c,c)}\]

Substituting Equation C-9 into Equation C-24 gives the final form of the equation defining the modal assurance criterion.

\[(C-25) \quad MAC(c,d) = \frac{|MOM(c,d)|^2}{MOM(c,c) \cdot MOM(d,d)}\]

The modal assurance criterion is a scalar constant representing the causal relationship between two modal vectors. The constant will take on values from zero, representing no consistent correspondence, to one, representing a consistent correspondence. In this manner, if the modal vectors under consideration truly exhibit a consistent relationship, the modal assurance criterion should approach unity and the value of the modal scale factor can be considered to be reasonable.

The terminology used in this derivation of the definition of modal scale factor and modal assurance criterion has not to this
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